RESISTANCE-BASED PERFORMANCE ANALYSIS OF THE CONSENSUS ALGORITHM OVER GEOMETRIC GRAPHS[*]

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Abstract. The performance of the linear consensus algorithm is studied by using a Linear Quadratic (LQ) cost. The objective is to understand how the communication topology influences this algorithm. This is achieved by exploiting the analogy between Markov chains and electrical resistive networks. Indeed, this allows to uncover the relation between the LQ performance cost and the average effective resistance of a suitable electrical network and, moreover, to show that, if the communication graph fulfils some local properties, then its behavior can be approximated by that of a grid, which is a graph whose associated LQ cost is well-known.

Key words. Multi-agent systems, consensus algorithm, distributed averaging, large-scale graphs

AMS subject classifications. 68R10, 90B10, 94C15, 90B18, 05C50

1. Introduction. The last two decades have witnessed a great effort within different scientific communities in the analysis of multi-agent systems. These systems, which consist of a large number of simple interacting entities, can be profitably used in order to model a number of rather different applications: just to recall some of them, load balancing [1], coordinated control [2], distributed estimation [3] and distributed calibration for sensor networks [4].

One of the basic problems to be solved when designing multi-agent systems is making the agents reach an agreement on a given quantity with limited communication effort. A popular tool to address this issue is the linear consensus algorithm, which is used when there is a set of agents, each with a scalar value, and the goal is to drive all agents to reach a common state, which is a weighted average of the initial values. The strength of this algorithm is mainly due to its simplicity and its intrinsic robustness to node/communication failures.

Mathematically, the consensus algorithm is a linear discrete-time dynamical system and its convergence can be analyzed with the same tools that are used for studying the convergence of Markov chains. Moreover, the performance analysis of this algorithm is usually based on the speed of its convergence, which coincides with the mixing rate of the associated Markov chain [5]. From a control-theoretic point of view, this way of evaluating the convergence of this algorithm coincides with the evaluation of a control system based on the allocation of eigenvalues. However, it is well-known in control that this is only one way to assess the performance of a dynamical system. Other ways are based of the capability of the systems to reject disturbances or to be robust to model uncertainty. These features can be mathematically quantified in terms of norms, specifically H^2 or H^∞ norms of some linear operators associated with

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the dynamical system.

In this paper we propose a performance analysis based on the 2-norm of the difference in time of the agents' states and the final consensus value. This provides in principle a different way to evaluate how fast the algorithm converges to its final value. This index is classical in control since it is the basis of the Linear Quadratic (LQ) optimal control design. It is well-known that this index describes also how a dynamic system attenuates the effect of a stochastic additive disturbance, and actually this is proved to hold true also for the linear consensus algorithm. We will show that this index is related to the effective resistance [6, 7] of the graph representing the allowed communication between the agents. The behavior of the index is known [8, 9, 10] in case of graphs which are regular grids over d-dimensional spaces. One of the main contributions of this paper is to show that the same behavior holds also for non-regular geometric graphs, as defined in [6, 11, 12], and hence that this behavior is related not to the graph regularity, but instead to its geometric nature. The results proposed in this paper were already described, without the proof details, in our paper [13] for the particular case in which the consensus matrix is assumed to be symmetric.

The paper is organized as follows. In Sect. 2 we give some basic notions on reversible consensus matrices. Sect. 3 is devoted to recalling the analogy among reversible consensus matrices and electrical networks, which allows us to state one of the main results. In Sect. 4 we show that the performance cost in a family of geometric graphs only depends on the graphs geometric dimension. The proofs of the results are postponed to Sect. 5 and Sect. 6. Finally, in Sect. 7 we draw some conclusions.

1.1. Notation. In this paper we will denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}_+$ the set of nonnegative real numbers. The symbol $\mathbb{R}^N$ will denote the vector space of N-dimensional column vectors with real entries. Vectors in $\mathbb{R}^N$ will be denoted by boldface letters, e.g. $\boldsymbol{x}$, while entries of vectors and scalars will be in italic font. We denote by $\mathbf{1}$ the column vector with all entries equal to 1 and of suitable dimension, and by $\boldsymbol{e}_u$ the u-th element of the canonical basis of $\mathbb{R}^N$, i.e., a vector whose u-th entry is 1 and all other entries are 0. Let $d_{\mathrm{E}}(\boldsymbol{u}, \boldsymbol{v})$ denote the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^N$. Given $\boldsymbol{v} \in \mathbb{R}^N$, the symbol $\operatorname{diag} \boldsymbol{v}$ denotes a $N \times N$ diagonal matrix whose (k, k)-th entry is the k-th entry of $\boldsymbol{v}$. Given $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^N$ and a positive definite matrix M, we will denote the inner product and the norm weighted by M as $\langle \boldsymbol{v}, \boldsymbol{w} \rangle_M := \boldsymbol{v}^T M \boldsymbol{w}$ and $\|\boldsymbol{v}\|_M := \sqrt{\boldsymbol{v}^T M \boldsymbol{v}}$, respectively.

1.2. Problem statement. Assume that the agents are labeled by the elements of the set $V = \{1, \dots, N\}$ and that the graph $\mathcal{G} = (V, \mathcal{E})$, where $\mathcal{E} \subseteq V \times V$, describes which communication links exist between the agents. More precisely, we say that $(u, v) \in \mathcal{E}$ if and only if the agent v sends information to u. The graph $\mathcal{G}$ is called the communication graph. In the linear consensus algorithm, at each iteration, the agents send their current state to their neighbors, and then update their states as a suitable convex combination of the received messages. More precisely, if $x_u(t)$ denotes the state of the agent $u \in V$ at time $t \in \mathbb{N}$, then

$$x_u(t+1) = \sum_{v \in V} P_{uv} x_v(t)\,, \tag{1.1}$$

where P_{uv} are the entries of a stochastic[1] matrix P. In a more compact form we can write

$$\boldsymbol{x}(t+1) = P\boldsymbol{x}(t) \tag{1.2}$$

[1] A matrix P is said to be stochastic if $P_{uv} \geq 0$ for all $u, v \in V$ and $\sum_{v \in V} P_{uv} = 1$ for all $u \in V$.

where $\boldsymbol{x}(t) \in \mathbb{R}^N$ denotes the vector collecting all agents' states.

The constraint imposed by the communication graph $\mathcal{G}$ is enforced by requiring that $\mathcal{G}_P = \mathcal{G}$, where $\mathcal{G}_P = (V, \mathcal{E}_P)$ is the *graph associated with the matrix* P, defined by letting $(u, v) \in \mathcal{E}_P$ if and only if $P_{uv} \neq 0$. Notice that we consider the equality $\mathcal{G}_P = \mathcal{G}$ for simplicity, while in principle we could require that $\mathcal{G}_P$ is a subgraph of $\mathcal{G}$ (i.e., that $\mathcal{E}_P \subseteq \mathcal{E}$), since a designer could choose not to use some of the available communication links.

The stochastic matrix P is said to be *irreducible* if the associated graph is strongly connected, namely, for all $u, v \in V$, there exists a path in $\mathcal{G}_P$ connecting u to v, and it is said to be *aperiodic* if the greatest common divisor of the lengths of all cycles in $\mathcal{G}_P$ is one. Notice that the presence of a self-loop, namely a $P_{uu} > 0$ for some $u \in V$, ensures aperiodicity.

The well-known Frobenius-Perron theory [5] ensures that, if P is irreducible and aperiodic, then: P has the eigenvalue 1 with algebraic multiplicity 1; the corresponding right eigenvector is $\mathbf{1}$ and the left eigenvector has strictly positive entries; such a left eigenvector, denoted by $\boldsymbol{\pi}^T$ and normalized so that $\sum_u \pi_u = 1$, is called the *invariant measure* of P. Moreover, under the same assumptions, all the other eigenvalues have absolute value strictly smaller than 1 and so we have that

$$P^t \overset{t\to\infty}{\longrightarrow} \mathbf{1}\boldsymbol{\pi}^T \,,$$

and consequently the states of the consensus algorithm (1.2) converge to the same value $x_u(t) \overset{t\to\infty}{\longrightarrow} \alpha$, where $\alpha = \boldsymbol{\pi}^T \boldsymbol{x}(0)$.

The convergence to the consensus value is exponential with exponent equal to the second largest eigenvalue of the matrix P

$$\rho(P) = \max\{|\lambda| : \lambda \in \sigma(P), \lambda \neq 1\}$$

where $\sigma(P)$ is the spectrum of P. For this reason, the value $\rho(P)$ is a classical performance cost for the algorithm. Indeed, the closer is $\rho(P)$ to zero, the faster is the algorithm. However, as recent papers have pointed out [10, 9, 14], this performance index is not the only possible choice for evaluating the performance of the algorithm. Different costs arise from different specific applications where the consensus algorithm is used. Moreover, it can be shown [14, 15] that, by considering different performance indices, it is possible to obtain different optimal graph topologies.

In this paper, we propose a Linear Quadratic (LQ) cost which is a performance index widely used in the control community. To evaluate how fast P^t converges to its limit value $\mathbf{1}\boldsymbol{\pi}^T$ we propose the index

$$J(P) := \frac{1}{N} \sum_{t\geq 0} ||P^t - \mathbf{1}\boldsymbol{\pi}^T||_F^2 = \frac{1}{N} \operatorname{trace}\left[\sum_{t\geq 0} (I - \boldsymbol{\pi}\mathbf{1}^T)(P^T)^t P^t (I - \mathbf{1}\boldsymbol{\pi}^T)\right],$$

where $\|M\|_{\mathrm{F}} := \sqrt{\operatorname{trace}(MM^T)}$ is the Frobenius norm of a matrix.

This cost appears in two different contexts. Assume first that we want to evaluate the speed of convergence of the consensus algorithm by the ℓ^2 norm of the transient, namely

$$\frac{1}{N} \sum_{t\geq 0} \left[\|\boldsymbol{x}(t) - \boldsymbol{x}(\infty)\|^2\right] .$$

Notice that this ℓ^2 norm will depend on the initial condition $\boldsymbol{x}(0)$. For this reason, we assume that the initial condition is a random variable with zero-mean and covariance matrix $\mathbb{E}\left[\boldsymbol{x}(0)\boldsymbol{x}(0)^T\right] = I$. We can now consider the expected value of the ℓ^2 norm of the transient which is now a function of the matrix P only. Indeed, by some simple computations [14] it can be shown that

$$\mathbb{E}\left[\frac{1}{N}\sum_{t\geq 0}\|\boldsymbol{x}(t)-\boldsymbol{x}(\infty)\|^2\right] = J(P)\,.$$

The cost $J(P)$ appears also in the context of noisy consensus [10, 14, 16]. Consider a network of agents implementing the consensus algorithm, in which the updates are affected by additive noise, so that the algorithm can be described by the following equation

$$\boldsymbol{x}(t+1) = P\boldsymbol{x}(t) + \boldsymbol{n}(t)\,,$$

where $\boldsymbol{n}(t)$ is a i.i.d. process. Assume that $\mathbb{E}[\boldsymbol{n}(t)] = 0$ and $\mathbb{E}[\boldsymbol{n}(t)\boldsymbol{n}(t)^T] = I$, the identity matrix of dimension $N\times N$, for all $t\in\mathbb{N}$. Assume that the initial condition $\boldsymbol{x}(0)$ is random and that it is uncorrelated from the noise process. We are interested in the dispersion of $\boldsymbol{x}(t)$. If we measure it by evaluating the displacement of $\boldsymbol{x}(t)$ from the weighted average $\sum_i \pi_i x_i(t)$, namely by introducing the vector

$$\boldsymbol{e}(t) = (I - \mathbf{1}\boldsymbol{\pi}^T)\boldsymbol{x}(t),$$

then it can be shown that

$$\frac{1}{N}\lim_{t\to\infty}\mathbb{E}\left[\|\boldsymbol{e}(t)\|^2\right] = J(P)\,.$$

Thus, the proposed LQ cost also characterizes the spreading of the asymptotic value of the state vector around its weighted average in a noisy network.

It is possible to consider the problem of determining the matrix P satisfying a constraint and minimizing the index $J(P)$. In this paper we will instead consider a different problem. Indeed we will try to provide estimates of $J(P)$ which permit to understand how this index depends on the structure of P and in particular on the topological properties of the graph $\mathcal{G}_P$ associated with P. We will be able to unveil this dependence by proving that $J(P)$ is related to the effective resistance of a suitable electrical network. This geometric parameter depends on the topology only, and not on the particular entries of P. Since the electrical analogy holds only if P is reversible [6], in this paper we will restrict to this class of matrices. An important subclass of reversible matrices consists of symmetric matrices, for which stronger and simpler results hold true [13].

Using these results we will show that, analogously to what happens for the convergence rate [17, 18], under some assumptions, a large class of graphs, called geometric and which can be seen as perturbed grids, exhibit a particular behavior of the cost $J(P)$ as a function of the number of nodes which depends on the geometric dimension of the graph. In particular, if the graph has geometric dimension one, namely it is a geometric graph on a segment, then $J(P)$ grows linearly in the number of nodes, while if the graph has geometric dimension two, namely it is a geometric graph on a square, then $J(P)$ grows logarithmically in the number of nodes. Finally, if the graph has geometric dimension three (or more), namely it is a geometric graph on a cube, then $J(P)$ is bounded from above by a constant independent of the number of nodes. This result is based on (and extends) an analogous result [8, 9, 10] which holds for torus graphs. In this way we show that the spatial invariance of torus graphs is not a necessary requirement for having this kind of behavior of $J(P)$.

2. Preliminaries. In this paper we will analyze the cost function $J(P)$ when P is an irreducible and aperiodic stochastic matrix, so that $J(P)$ is finite. In fact, aperiodicity will be the consequence of the stronger assumption which imposes that the diagonal elements of P are all positive. Observe that this condition is not restrictive for the consensus algorithm as it assumes only that in the state-update (1.1) each agent gives to its own current state a positive weight and this does not require additional communication. For this reason, throughout the paper we will use the following definition.

DEFINITION 2.0.1. *We say that a matrix P is a* consensus matrix *if it is stochastic and irreducible, and it satisfies $P_{uu} > 0$ for all u.*

Recall [5] that a consensus matrix has a dominant eigenvalue 1 with algebraic multiplicity 1. As already pointed out, the corresponding left eigenvector $\boldsymbol{\pi}$, normalized so that $\sum_u \pi_u = 1$, has all entries which are strictly positive and is called the *invariant measure* of P, a name coming from the interpretation of P as the transition probability matrix of a Markov chain.

An useful object is the Laplacian of a matrix P, which is defined as $L := I - P \in \mathbb{R}^{N\times N}$. It is immediate to check that $L\mathbf{1} = \mathbf{0}$ and $L_{uv} \leq 0$, $\forall\, u \neq v$. Moreover, since P has 1 as eigenvalue with algebraic multiplicity one, then $\dim(\ker L) = 1$. One of the most important classes of consensus matrices is that of *reversible* matrices, a name which comes, again, from the definition of reversible Markov chains [19].

DEFINITION 2.0.2. *Let P be a consensus matrix with invariant measure $\boldsymbol{\pi}$. Then P is said to be reversible if*

$$\pi_u P_{uv} = \pi_v P_{vu},\ \forall\,(u,\, v) \in V \times V$$

or equivalently if

$$\Pi P = P^T \Pi, \tag{2.1}$$

where $\Pi = \operatorname{diag} \boldsymbol{\pi}$.

REMARK 2.0.1. *In general, the graph associated with a consensus matrix P is a directed graph. However, the assumption that the consensus matrix is reversible implies that the associated graph is undirected, i.e., $(u,\, v) \in \mathcal{E}_P \iff (v,\, u) \in \mathcal{E}_P$. The proof comes immediately from Eq.* (2.1) *and the property that $\pi_u > 0$ for all u, implying that $P_{uv} > 0$ if and only if $P_{vu} > 0$.*

For this reason, in the following we will always assume that the graph $\mathcal{G}_P$ is undirected. Also recall that in our definition a consensus matrix has non-zero diagonal elements, and thus we will consider only graphs having a self-loop at each node. However, consistently with previous literature of consensus, we will define the neighborhood of a node $u \in \mathcal{G}_P$ to be the set of its neighbors, except u itself

$$N_u := \{v \in V : v \neq u,\, (u,v) \in \mathcal{E}_P\}$$

and we will define the degree of u to be the cardinality $|N_u|$ of its neighborhood.

Notice that Eq. (2.1) states that reversible matrices are self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_\Pi$, or, equivalently, that the matrix $\Pi^{1/2} P \Pi^{-1/2}$ is symmetric. Hence $\Pi^{1/2} P \Pi^{-1/2}$ has real eigenvalues and N orthogonal eigenvectors, and this implies that P has real eigenvalues too, and N independent eigenvectors.

Now we briefly recall the notion of Green matrix of a consensus matrix P, which is also known as fundamental matrix in the Markov chains literature. Here we concentrate only on the results that are needed in the paper. A more extensive list of properties of the fundamental matrix can be found in [20].

DEFINITION 2.0.3. *Let P be a consensus matrix, with invariant measure $\boldsymbol{\pi}$. The Green matrix G of P is defined as*

$$G := \sum_{t\geq 0}(P^t - \mathbf{1}\boldsymbol{\pi}^T)\,. \tag{2.2}$$

The Green matrix plays a fundamental role in this paper due to its property of being almost an inverse of the Laplacian, in the sense that

$$G + \mathbf{1}\boldsymbol{\pi}^T = (L + \mathbf{1}\boldsymbol{\pi}^T)^{-1}.$$

The above expression is easy to verify, and implies the following equation, which will be useful later on

$$\begin{bmatrix} G & \mathbf{1} \end{bmatrix} \begin{bmatrix} L \\ \boldsymbol{\pi}^T \end{bmatrix} = I\,. \tag{2.3}$$

3. Reversible consensus matrices and electrical networks. In this section we present electrical networks, their relation with consensus matrices, and the concept of effective resistance. Making use of these notions, we state our main results, Theorem 3.2.1 and Theorem 3.2.2, which give useful bounds of the cost $J(P)$ of a reversible consensus matrix P.

3.1. The electrical analogy. The analogy between consensus matrices, or Markov chains, and resistive electrical networks dates back to the work of Doyle and Snell [6]. It is a powerful tool which gives a strong intuition on the behavior of the chain on the basis of the physics of electrical networks, as well as permitting simple and clear proofs for many results. Our interest is mainly related to the possibility to rewrite the LQ cost introduced above in terms of a geometric parameter, the average effective resistance. To this respect we are strongly indebted in terms of inspiration to the papers by Barooah and Hespanha [21, 22, 12], from which we took many results we state here without a proof. Effective resistances also arise as a performance metric for clock synchronization algorithms in [23, 24], and methods for its minimization are proposed in [7]. To conclude, in [25] the effective resistance is computed in terms of the eigenvalues of the Laplacian matrix in the symmetric case.

3.1.1. Electrical networks. A resistive electrical network is a graph in which pairs of nodes are connected by resistors. A resistive electrical network is therefore determined by a symmetric matrix C with non-negative entries which tells for each pair of nodes u, v which is the conductance of the resistor connecting those two nodes. A resistive electrical network is said to be connected if the graph associated with C is connected.

In order to describe the current flowing on the electrical network and to write Kirchhoff's and Ohm's laws, we choose (arbitrarily) a conventional orientation for each edge of the undirected graph $\mathcal{G}$, so that current will be denoted as positive when flowing consistently with the direction of the edge and negative otherwise. To this aim, for any pair of nodes u, v, such that $u \neq v$ and $C_{uv} = C_{vu} \neq 0$, we choose either (u, v) or (v, u) in $V \times V$. Let $\mathcal{E} \subseteq V \times V$ be the set of directed edges formed in this way and let M be the number of edges. Define the incidence matrix $B \in \mathbb{R}^{M\times N}$ as follows: order the edges from 1 to M and let for any $e \in \{1, \dots, M\}$ and $u \in \{1, \dots, N\}$

$$B_{eu} = \begin{cases} -1 & \text{if the edge } e \text{ is } (v,\, u) \text{ for some } v \neq u, \\ 1 & \text{if the edge } e \text{ is } (u,\, v) \text{ for some } v \neq u, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

We define the diagonal matrix $\mathcal{C} \in \mathbb{R}^{M\times M}$ having the conductance of the edge e as the entry in position $(e,\, e)$. The relation between the matrices $\mathcal{C}$ and C is easily obtained as

$$B^T \mathcal{C} B = \operatorname{diag} C\mathbf{1} - C\,, \tag{3.2}$$

namely

$$[B^T \mathcal{C} B]_{uv} = \begin{cases} C_u & \text{if } u = v, \\ -C_{uv} & \text{if } (u,\, v) \in \mathcal{E}, \\ 0 & \text{if } (u,\, v) \notin \mathcal{E}, \end{cases}$$

where $C_u := \sum_{v\in V} C_{uv}$.

Let $\boldsymbol{i} \in \mathbb{R}^N$ be given such that $\boldsymbol{i}^T \mathbf{1} = 0$, and interpret the k-th entry of $\boldsymbol{i}$ as the current which is injected (or extracted if negative in sign) into the k-th node of the network from an external source. The currents $\boldsymbol{i}$ induce current flows on edges and potentials at the nodes, which we denote by $\boldsymbol{j} \in \mathbb{R}^M$ and $\boldsymbol{v} \in \mathbb{R}^N$, respectively. Currents and potentials are related through the well–known Kirchhoff's node law and Ohm's law, which can be expressed as the following system of linear equations

$$\begin{cases} B^T \boldsymbol{j} = \boldsymbol{i} \\ \mathcal{C} B \boldsymbol{v} = \boldsymbol{j} \end{cases} \tag{3.3}$$

where the first equation states that the total current flow entering into each node equals the total flow exiting from it (Kirchhoff's current law), while the second equation represents Ohm's law, $C_{uu'}(v_u - v_{u'}) = j_e$ for all $e = (u,\, u') \in \mathcal{E}$.

Solving the electrical network means finding the solutions $\boldsymbol{j}$ and $\boldsymbol{v}$ of Eq. (3.3), in particular finding the solution $\boldsymbol{v}$ of the following equation

$$B^T \mathcal{C} B \boldsymbol{v} = \boldsymbol{i}\,. \tag{3.4}$$

It is well known (see, e.g., [6]) that a solution of the previous equation exists and, for a connected network, that it is unique up to a constant additive term for $\boldsymbol{v}$, i.e., the differences $v_u - v_{u'}$ are uniquely defined. In the next subsection, we will give an explicit expression for the solutions $\boldsymbol{v}$, involving the Green matrix of the associated reversible consensus matrix.

Given a connected electrical network with conductance matrix C, the *effective resistance* between two nodes u, u' is defined to be

$$\mathcal{R}_{uu'}(C) := v_u - v_{u'}$$

where we impose $\boldsymbol{i} = \boldsymbol{e}_u - \boldsymbol{e}_{u'}$ and $\boldsymbol{v}$ is any solution of the corresponding electrical equation (3.4), namely, $\boldsymbol{v}$ is the potential at each node of the network in the case a unit current is injected at node u and extracted from node u'. Finally the *average effective resistance* of the electrical network is defined as

$$\bar{\mathcal{R}}(C) := \frac{1}{2N^2} \sum_{u,\, u' \in V} \mathcal{R}_{uu'}(C). \tag{3.5}$$

Given a connected undirected graph $\mathcal{G}$, in the following we will use the notations $\mathcal{R}_{uu'}(\mathcal{G})$ and $\bar{\mathcal{R}}(\mathcal{G})$, as the effective resistance and the average effective resistance associated with the electrical network having conductance equal to 1 for all the edges of $\mathcal{G}$ and conductance equal to 0 otherwise.

3.1.2. Electrical network associated with a consensus matrix. There is a way to obtain a one to one relation between reversible consensus matrices and connected resistive electrical networks with some fixed total conductance (i.e., sum of the conductances of all edges). Let P be a reversible consensus matrix and let

$$\Phi(P) := N\Pi P$$

where $\Pi = \operatorname{diag}(\boldsymbol{\pi})$ and $\boldsymbol{\pi}$ is the invariant measure of P. It is clear that $\Phi(P)$ is the conductance matrix of a connected resistive network. It can be shown that the map Φ is injective. Indeed, if P_1, P_2 are reversible consensus matrices and if $\Phi(P_1) = \Phi(P_2)$, then $\operatorname{diag}(\boldsymbol{\pi}_1)P_1 = \operatorname{diag}(\boldsymbol{\pi}_2)P_2$. Multiplying on the right both members by $\mathbf{1}$ we obtain that $\boldsymbol{\pi}_1 = \boldsymbol{\pi}_2$ and consequently $P_1 = P_2$. We show now that $\operatorname{Im}(\Phi) = \mathcal{S}$, where

$$\mathcal{S} := \{C \in \mathbb{R}_+^{N\times N} : C = C^T,\, C_{uu} > 0\ \forall\, u \in V,\, \mathcal{G}_C \text{ is connected, and } \mathbf{1}^T C \mathbf{1} = N\}$$

Clearly $\operatorname{Im}(\Phi) \subseteq \mathcal{S}$. To prove the equality, for any given $C \in \mathcal{S}$ we consider

$$P = (\operatorname{diag}(C\mathbf{1}))^{-1}\, C \tag{3.6}$$

and we show that P is a reversible consensus matrix such that $\Phi(P) = C$. It is straightforward to see that $P\mathbf{1} = \mathbf{1}$, $P_{uu} > 0,\ \forall\, u \in V$, and that $\mathcal{G}_P$ is connected.

To prove that P is reversible and that $\Phi(P) = P$, the key remark is that

$$\boldsymbol{\pi} = \frac{1}{N} C\mathbf{1}$$

is the invariant measure of P, and thus $\Pi = \frac{1}{N}\operatorname{diag}(C\mathbf{1})$. This immediately implies that $\Phi(P) = C$. The reversibility of P is then proved by using the symmetry of C: $\Pi P = \frac{1}{N} C = \frac{1}{N} C^T = P^T \Pi$. In this way, we have proved not only that Φ is bijective on $\mathcal{S}$, but also that (3.6) provides the inverse of Φ over $\mathcal{S}$.

Now, consider a reversible consensus matrix P and its associated conductance matrix $C := \Phi(P)$. Let moreover B and $\mathcal{C}$ be the matrices associated with the resistive electrical network with conductance C, as defined above. Notice that the Laplacian matrix $L := I - P = \frac{1}{N}\Pi^{-1}B^T\mathcal{C}B$ and so the equation 3.4 is equivalent to

$$L\boldsymbol{v} = \tfrac{1}{N}\Pi^{-1}\boldsymbol{i}. \tag{3.7}$$

Since the network is connected, $\ker L = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$, and thus, for any $\boldsymbol{i}$ such that $\boldsymbol{i}^T\mathbf{1} = 0$, Eq. (3.7) has infinitely many solutions, of the form $\boldsymbol{v} + \alpha\mathbf{1}$ for some real constant α, where $\boldsymbol{v}$ is a particular solution. In our setting it is convenient to find $\boldsymbol{v}$ which satisfies the following constraint

$$\boldsymbol{\pi}^T\boldsymbol{v} = 0,$$

which means that we need to solve the equation

$$\begin{bmatrix} L \\ \boldsymbol{\pi}^T \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} \frac{1}{N}\Pi^{-1}\boldsymbol{i} \\ 0 \end{bmatrix}.$$

Thanks to Eq. (2.3), the solution can be obtained by using the Green matrix G associated with P, as follows

$$\boldsymbol{v} = \begin{bmatrix} G & \mathbf{1} \end{bmatrix} \begin{bmatrix} \frac{1}{N}\Pi^{-1}\boldsymbol{i} \\ 0 \end{bmatrix} = \tfrac{1}{N} G\Pi^{-1}\boldsymbol{i}.$$

Consequently, we can obtain the effective resistance as follows

$$\mathcal{R}_{uu'}(C) = \tfrac{1}{N}(\boldsymbol{e}_u - \boldsymbol{e}_{u'})^T G\Pi^{-1}(\boldsymbol{e}_u - \boldsymbol{e}_{u'}). \tag{3.8}$$

3.2. LQ cost and effective resistance. This section is devoted to our main results on the relation between the LQ cost $J(P)$ for a reversible consensus matrix P and the average effective resistance of a suitable electrical network. The results are then formulated in the special case of symmetric consensus matrices, since for this case they turns out to be clearer and more readable. The proofs are given in Sect. 5.

Consider a reversible consensus matrix P and let $\boldsymbol{\pi}$ be its invariant measure. Build the electrical network associated with the matrix P^2 as suggested in Sect. 3.1.2, namely build the matrix of conductances

$$C := \Phi(P^2) = N\Pi P^2. \tag{3.9}$$

In the particular case in which P is symmetric we have that $C = P^2$. The following theorem allows us to estimate the cost in terms of the average effective resistance of this electrical network, and of quantities depending on the elements of the invariant measure of P.

THEOREM 3.2.1. *Let $P \in \mathbb{R}^{N\times N}$ be a reversible consensus matrix and $\boldsymbol{\pi}$ its invariant measure, and C the matrix of conductances defined in Eq. (3.9). Then it holds*

$$\frac{\pi_{\min}^3 N^2}{\pi_{\max}} \bar{\mathcal{R}}(C) \leq J(P) \leq \frac{\pi_{\max}^3 N^2}{\pi_{\min}} \bar{\mathcal{R}}(C),$$

where $\pi_{\min}$ and $\pi_{\max}$ are respectively the minimum and maximum entries of $\boldsymbol{\pi}$.

In the particular case of a symmetric matrix we have the following corollary which is a straightforward consequence of the previous theorem.

COROLLARY 3.2.1. *Let $P \in \mathbb{R}^{N\times N}$ be a symmetric consensus matrix, and C the matrix of conductances defined in Eq. (3.9). Then it holds*

$$J(P) = \bar{\mathcal{R}}(C).$$

The previous results involve the electrical network built from the matrix P^2. The following theorem, instead, describes the behavior of $J(P)$ in terms of $\bar{R}(\mathcal{G})$, namely the average effective resistance of an electrical network obtained from the graph $\mathcal{G}$ by associating a unit resistance to each edge. This result underlines the role of the graph, irrespective of the particular entries of the matrix P.

THEOREM 3.2.2. *Let P be a reversible consensus matrix with invariant measure $\boldsymbol{\pi}$ and let $\mathcal{G}$ be the graph associated with P. Assume that all the non-zero entries of P*

belong to the interval $[p_{\min}, p_{\max}]$*, and that the degree of any node is bounded from above by an integer* δ*. Then,*

$$\frac{\pi_{\min}^3 N}{8p_{\max}^2\delta^2\pi_{\max}^2}\bar{\mathcal{R}}(\mathcal{G}) \leq J(P) \leq \frac{\pi_{\max}^3 N}{p_{\min}^2\pi_{\min}^2}\bar{\mathcal{R}}(\mathcal{G})\,.$$

A simpler result can be obtained for symmetric matrices as a straightforward consequence of the previous theorem.

COROLLARY 3.2.2. *Let* P *be a symmetric consensus matrix associated with a graph* $\mathcal{G}$*. Assume that all the non-zero entries of* P *belong to the interval* $[p_{\min}, p_{\max}]$*, and that the degree of any node is bounded from above by an integer* δ*. Then,*

$$\frac{1}{8p_{\max}^2\delta^2}\bar{\mathcal{R}}(\mathcal{G}) \leq J(P) \leq \frac{1}{p_{\min}^2}\bar{\mathcal{R}}(\mathcal{G})\,.$$

These last two results can be used to estimate the proposed LQ-cost in terms of the effective resistance of graphs only, as we will show in Sect. 4.

3.3. LQ cost and network dimension. One of the most important problems in the design of a sensor network is the choice of its size, namely the decision of how many sensors to deploy in order to obtain a given performance. From this point of view, it is very important to understand how our cost function scales in terms of the number N of nodes in a sequence of graphs of growing size, belonging to a given family. The results in the previous sections can be used to achieve this goal. More precisely, consider a sequence $\{\mathcal{G}_N\}$ of graphs, and a sequence $\{P_N\}$ of reversible consensus matrices associated with such graphs, and consider the problem of understanding the asymptotic behavior of the cost $J(P_N)$ when N tends to infinity. Theorem 3.2.2 shows that the asymptotics of $J(P_N)$ are intimately related to those of $\bar{\mathcal{R}}(\mathcal{G}_N)$. However, the terms δ, $p_{\min}$, $p_{\max}$, $\pi_{\min}$ and $\pi_{\max}$ appearing in Theorem 3.2.2 have their own influence on the asymptotics. The result in Theorem 3.2.2 is particularly interesting in all the cases when the family under study satisfies the following assumptions:

1. there is an upper bound δ for all maximum degrees of the graphs in the family;
2. there is a lower bound $p_{\min}$ and an upper bound $p_{\max}$ for all the non-zero entries in all matrices P_N of the family;
3. there exist two constants c_l and c_u such that, for all N, $\frac{c_l}{N} \leq \pi_{N,\min} \leq \pi_{N,\max} \leq \frac{c_u}{N}$, where $\pi_{N,\min}$ and $\pi_{N,\max}$ denote the minimum and maximum values of $\boldsymbol{\pi}_N$, the invariant measure of P_N.

Indeed, under such assumptions the asymptotic behavior of the cost $J(P_N)$ is completely described (up to multiplicative constants) by the behavior of $\bar{\mathcal{R}}(\mathcal{G}_N)$.

The first two assumptions are quite natural when the goal is to understand the effect of local interactions, and are easy to verify. The assumption on the asymptotics of the invariant measure deserves more attention. A first important remark is that, as it is underlined in Corollary 3.2.2, this assumption is satisfied whenever we consider symmetric matrices P_N. Indeed, we can obtain symmetric consensus matrices in case we start from undirected graphs and we use the most popular consensus protocol

$$x_u(t+1) = x_u(t) + \varepsilon \sum_{v\in N_u}(x_v(t) - x_u(t)) \tag{3.10}$$

where ε is a small enough positive constant.

If we relax the assumption that all P_N's are symmetric, and we consider a more general family of reversible matrices P_N, then the assumption that the sequences $N\pi_{N,\,\min}$ and $N\pi_{N,\,\max}$ are uniformly bounded from above and below by constants independent of N is not trivial any more, and is not implied, in general, by the two assumptions on the degree and on the minimum and maximum coefficients. However this is the case for some notable and popular protocols. For example, consider the following consensus iteration (known as "simple random walk" or "uniform weights")

$$x_u(t+1) = \frac{1}{|N_u|+1}\Big(x_u(t) + \sum_{v\in N_u} x_v(t)\Big). \tag{3.11}$$

It can be shown that, if the graph is undirected, the corresponding non-symmetric consensus matrix is reversible, and that the invariant measure $\boldsymbol{\pi}$ has components

$$\pi_u = \frac{|N_u|+1}{\sum_{v\in V}(|N_v|+1)}.$$

Under the assumption that the degree of any node is bounded from above by an integer δ, it is clear that the entries of the consensus matrix belong to the interval $[\frac{1}{\delta+1},\,\frac{1}{2}]$ and that each entry of the invariant measure lies in the interval $[\frac{2}{(\delta+1)N},\,\frac{\delta+1}{2N}]$. Consequently the assumptions above are satisfied.

On the contrary, the following example shows a family of reversible consensus matrices P_N whose entries belong to a fixed interval $[p_{\min},\,p_{\max}]$ and whose associated graph has uniformly bounded degrees, but for which $N\pi_{N,\,\min}$ and $N\pi_{N,\,\max}$ are not uniformly bounded. The example is depicted in Figure 3.1. One can check that the corresponding consensus matrix is reversible and that the invariant measure $\boldsymbol{\pi}$ is such that $\pi_k = \alpha(a/b)^{k-1}, k = 1,\ldots,N$, where α is a suitable normalizing factor. If we assume that $a > b$, then $\pi_{N,\,\min} = \pi_1 = \alpha$ and $\pi_{N,\,\max} = \pi_N = \alpha(a/b)^{N-1}$. In this case $N\pi_{N,\,\min}$ and $N\pi_{N,\,\max}$ cannot be uniformly bounded from below and above, because if this were the case, then also the ratio $N\pi_{N,\,\max}/N\pi_{N,\,\min}$ would be uniformly bounded from below and above, but this is not possible, as $N\pi_{N,\,\max}/N\pi_{N,\,\min} = \pi_{N,\,\max}/\pi_{N,\,\min} = (a/b)^{N-1}$.

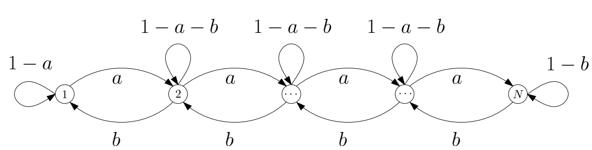

Figure 3.1. *Example: a family of growing lines. We assume that $a > 0$, $b > 0$ and $a + b < 1$.*

4. Application to geometric graphs.

4.1. Geometric graphs: definition. Let $\mathcal{G} = (V,\,\mathcal{E})$ be a connected undirected graph such that $V \subset Q$ where $Q = [0,\ell]^d \subseteq \mathbb{R}^d$ and $|V| = N$. Namely, the nodes of the graph are deployed in some d-dimensional hypercube of side length equal to ℓ. Such a graph will be called a *geometric graph*. Following [22], we define the following parameters associated with a geometric graph $\mathcal{G}$:

- the minimum Euclidean distance between any two nodes

$$s = \min_{u,\,v\in V,\,u\neq v}\{d_{\mathrm{E}}(u,\,v)\}; \tag{4.1}$$

- the maximum Euclidean distance between any two connected nodes

$$r = \max_{(u,\,v)\in\mathcal{E}} \{d_{\mathrm{E}}(u,\,v)\}\,; \tag{4.2}$$

- the radius of the largest ball centered in Q not containing any node of the graph

$$\gamma = \sup\left\{r > 0 |\, B(\boldsymbol{x},\,r) \cap V = \emptyset,\ \text{for some}\ \boldsymbol{x} \in Q\right\}, \tag{4.3}$$

  where $B(\boldsymbol{x},\,r)$ denotes the d-dimensional ball centered in $\boldsymbol{x} \in \mathbb{R}^d$ and with radius r;
- the minimum ratio between the Euclidean distance of two nodes and their graphical distance[2]

$$\rho = \min\left\{\frac{d_{\mathrm{E}}(u,\,v)}{d_{\mathcal{G}}(u,\,v)} |\, u,\,v \in V,\, u \neq v\right\}. \tag{4.4}$$

REMARK 4.1.1. *It is important to notice that the parameters s and γ depend only on the positions of the points and not on the graph itself, while r and ρ depend both on the positions and on the edge set. Moreover, the parameters are not independent of each other. Indeed, it is easy to see that they satisfy the following inequalities:*

$$s \leq r\,,\ \ s \leq 2\gamma\,,\ \ \rho \leq r\,,$$

where the last inequality follows from the fact that, given any pair of neighbor nodes $\bar{u}, \bar{v}$, then

$$\rho \leq \frac{d_{\mathrm{E}}(\bar{u},\,\bar{v})}{d_{\mathcal{G}}(\bar{u},\,\bar{v})} = \frac{d_{\mathrm{E}}(\bar{u},\,\bar{v})}{1} \leq r\,.$$

Finally, the degree of each node is bounded by the ratio of the volume of two spheres, with radius r and s respectively, and hence the maximum degree δ is bounded as follows:

$$\delta \leq \frac{r^d}{s^d}\,.$$

Clearly one graph can give rise to different geometric graphs, depending on how its vertices are deployed in the Euclidean space, and the parameters s, r, γ and ρ will depend on this deployment. Our results, and in particular Theorem 4.2.1, hold true for any choice of the representation of the graph as a geometric graph, although different choices will lead to tighter or looser bounds.

Motivated by the study of sensor networks, we are particularly interested in the cases in which there is a natural choice of the Euclidean space and of the way the vertices of the graph are deployed in it, since these vertices represent sensors distributed over a region. Moreover, in many sensor networks the communication graph is completely determined by the positions of the sensors. These situations are well described by the concept of *proximity graphs* [2], which are geometric graphs for which the edge set $E \subseteq V \times V$ is a function of V, namely on the positions of the nodes only.

[2]The graphical distance $d_{\mathcal{G}}(u,\,v)$ between u and v is defined as the length (i.e., number of edges) of the minimum path connecting u and v.

A simple example of proximity graph is the R-disk graph, where ad edge (u, v) exists if and only if the Euclidean distance between u and v is at most R. A particular case of an R-disk graph is the *regular grid* with dimension d. This is defined as the geometric graph with $N = n^d$ nodes, lying in a hypercube of edge length $\ell = n - 1$, whose nodes have coordinates $(i_1, \dots, i_d)$, where $i_1, \dots, i_d \in \{0, 1, \dots, n-1\}$, and in which two nodes u, v are connected if and only if $d_{\mathrm{E}}(u, v) \le 1$. For this graph, the geometric parameters can be computed explicitly: $s = r = 1$, $\gamma = 1/\sqrt{2}$ and $\rho = 1/2$.

Another example of proximity graph is the Delaunay graph, in which two vertices are connected if and only if the associated Voronoi cells are adjacent.

4.2. Geometric graphs: main result. The following theorem, whose proof can be found in Sect. 6, is the main result of this section.

THEOREM 4.2.1. *Let $P \in \mathbb{R}^{N \times N}$ be a reversible consensus matrix with invariant measure $\boldsymbol{\pi}$, associated with a graph $\mathcal{G} = (V, \mathcal{E})$. Assume that all the non-zero entries of P belong to the interval $[p_{\min}, p_{\max}]$ and that $\mathcal{G}$ is a geometric graph with parameters (s, r, γ, ρ) and nodes lying in $Q = [0, \ell]^d$ in which $\gamma < \ell/4$. Then*

$$k_1 + q_1 f_d(N) \le J(P) \le k_2 + q_2 f_d(N)\,,$$

where

$$f_d(N) = \begin{cases} N & \text{if } d = 1, \\ \log N & \text{if } d = 2, \\ 1 & \text{if } d \ge 3, \end{cases} \tag{4.5}$$

and where k_1, k_2, q_1 and q_2 are positive numbers which are functions of the following parameters only: the dimension d, the geometric parameters of the graph (s, r, γ, and ρ), the maximum degree δ, $p_{\min}$ and $p_{\max}$ and the products $\pi_{\min} N$ and $\pi_{\max} N$.

In the next section we will show many interesting families of geometric graphs and of associated consensus matrices in which the function $f_d(N)$ captures the dependence of $J(P_N)$ on the graph size N. This is not trivially true, since the statement of the previous theorem only has the advantage of separating $f_d(N)$, an explicit function of N, from other terms which exhibit no direct dependence on N, but may depend on N indirectly through the other parameters.

4.3. Families of geometric graphs with bounded parameters. In Sect. 3.3, we have discussed how Theorem 3.2.2 can be used to study the asymptotic behavior of the cost $J(P_N)$ for families of matrices P_N with growing size N, under some assumptions. Similarly, here we will study the asymptotic behavior for families of geometric graphs, showing that in many interesting cases the asymptotics of $J(P_N)$ is completely described by the above-defined function $f_d(N)$, thanks to Theorem 4.2.1.

Consider a growing family of geometric graphs $\mathcal{G}_N$ with $\mathcal{G}_N = (V_N, \mathcal{E}_N)$ and $|V_N| = N$, $V_N \subset [0, \ell_N]^d$. Let each $\mathcal{G}_N$ be a geometric graph with parameters s_N, r_N, γ_N and ρ_N. We will say that this is a *family of geometric graphs with bounded parameters* if there exist four positive constants s, r, γ and ρ, which we call *the geometric parameters of the family*, such that

$$s_N \ge s\,,\ r_N \le r\,,\ \gamma_N \le \gamma\,,\ \rho_N \ge \rho\,, \quad \forall N\,. \tag{4.6}$$

By Remark 4.1.1, such inequalities are enough to ensure that s_N, r_N, γ_N and ρ_N are bounded from below and from above. Indeed, the bounds in (4.6) imply the following bounds

$$s \le s_N \le r_N \le r\,,\ \rho \le \rho_N \le r\,,\ \frac{s}{2} \le \gamma_N \le \gamma\,.$$

Moreover, the maximum degree δ_N is bounded as follows

$$1 \le \delta_N \le \frac{r^d}{s^d}\,, \quad \forall\, N$$

where the lower bound is true under the assumption that all graphs of the family are connected.

Consider a family of connected geometric graphs with bounded parameters s, r, γ and ρ, and consider an associated family of reversible consensus matrices. Let P_N be a reversible consensus matrix associated with $\mathcal{G}_N$, and let $\boldsymbol{\pi}_N$ be its invariant measure. Assume that there exist four positive constants $p_{\min}$, $p_{\max}$, c_l and c_u such that, for all N, all non-zero entries of P_N belong to the interval $[p_{\min}, p_{\max}]$, and the invariant measures satisfies $N\pi_{N,\min} \ge c_l$ and $N\pi_{N,\max} \le c_u$. Then, Theorem 4.2.1 ensures that

$$k_1 + q_1 f_d(N) \le J(P_N) \le k_2 + q_2 f_d(N)\,, \quad \forall N\,,$$

where k_1, k_2, q_1 and q_2 are positive constants, which are independent of N, and are a function of the following parameters only: the dimension d, the geometric parameters of the graph family (s, r, γ, and ρ), the four constants $p_{\min}$, $p_{\max}$, c_l and c_u.

The assumptions involving the non-zero entries of P_N and the invariant measure have been discussed in detail in Sect. 3.3. Recall in particular that in case P_N are symmetric consensus matrices, then the assumption $N\pi_{N,\min} \ge c_l$ and $N\pi_{N,\max} \le c_u$ are satisfied with $c_l = c_u = 1$. In the following, we will show some examples of families of geometric graphs with bounded geometric parameters. A prototypical example is the regular grid introduced in Sect. 4.1. In this family, each $\mathcal{G}_N$ has exactly the same geometric parameters $s = r = 1$, $\gamma = 1/\sqrt{2}$ and $\rho = 1/2$, which are thus the parameters of the family. Notice that the edge length of the cube is $\ell_N \sim \sqrt[d]{N}$ for $N \to \infty$: this is not specific of the grid only, but (up to a multiplicative constant) it is a property of all families of graphs with bounded geometric parameters in $\mathbb{R}^d$, as we will show in Lemma 6.0.7. The asymptotic behavior of $J(P_N)$ for the simple random walk over the regular grid is well-known [8, 9, 10], but the results were proved exploiting the regular structure (spatial invariance) of the grid and the resulting algebraic properties of P_N. Our results show that the same asymptotic behavior happens in more general grid-like graphs, for example obtained by perturbations of grids. A similar fact had been noticed for a different performance index, namely the rate of convergence (second largest eigenvalue of P_N) [17, 18].

The examples of proximity graphs introduced in Sect. 4.1 can provide other examples of families of geometric graphs with bounded parameters.

For the 2-dimensional Delaunay graphs, it is interesting to see that, in order to enforce the inequalities (4.6), it is sufficient to consider the first two assumptions only, namely $s_N \ge s$ and $\gamma_N \le \gamma$. Indeed, it is easy to show that $r_N \le 2\gamma_N$, and it has been proved [26] that $\rho_N \ge 2s_N/\pi$.

For the R-disk graph, it is immediate to see that $r_N \le R$. However, in addition to the two assumptions $s_N \ge s$ and $\gamma_N \le \gamma$ one still needs to assume that ρ_N is bounded away from zero, since there exist R-disk graphs with vanishing ρ_N. For example, the family of line graphs (regular grids with dimension 1) is a family of R-disk graphs with bounded parameters. However, by a different choice of points, the same graphs can be associated to a different family of R-disk geometric graphs in $\mathbb{R}^2$, illustrated in Figure 4.1, having vanishing ρ_N.

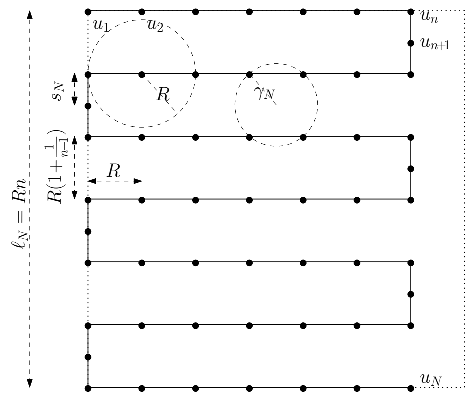

FIGURE 4.1. *Example of 2-dimensional R-disk graph with* $\lim_{N\to\infty} \rho_N = 0$. *It has* $N = n^2+n-1$ *vertices lying in a square of side* $\ell_N = Rn$. *Its geometric parameters are:* $s_N = \frac{R}{2}(1+\frac{1}{n-1}) > \frac{R}{2}$, $r_N = R$, $\gamma_N = \frac{R}{2}\sqrt{\left[\left(1+\frac{1}{n-1}\right)^2+1\right]} < \frac{5R}{2}$, *and* $\rho_N \leq \frac{d_{\mathrm{E}}(u_1,u_N)}{d_{\mathcal{G}}(u_1,u_N)} \leq \frac{R(n-1)\sqrt{2}}{N-1} \leq \frac{R\sqrt{2}}{\sqrt{N}}$.

In the next section we will illustrate the scaling result discussed above, with some simulations involving R-disk graphs where the positions of the points are chosen randomly.

4.4. Numerical results. In this section we propose some numerical experiments for a class of geometric graphs based on random geometric graphs [17]. The algorithm used to build these graphs is the following

- we fix the dimension d, the geometric parameters of the family $\bar{s}$, $\bar{r}$, $\bar{\gamma}$ and $\bar{\rho}$, an edge probability p_e and a positive constant c;
- we fix the number of nodes N and we build the hypercube $Q = [0,\ell]^d$, where $\ell := cN^{1/d}$;
- the set of nodes is picked randomly using the following iterative rule.
  - Initialization: the first node is picked randomly uniformly in Q;
  - Iterative step: the next node is picked randomly uniformly in Q, independently of previous nodes; if the Euclidean distance between the new node and any of the previous ones is less than $\bar{s}$, then discard the new node; repeat until having constructed N acceptable nodes;
- the set of edges is constructed as follows: among any pair of nodes whose relative Euclidean distance is smaller than $\bar{r}$, there is an edge with probability p_e (independently of other edges);
- the graph is discarded if it is not connected;
- we compute the geometric parameters γ and ρ, and we discard the graph if $\gamma > \bar{\gamma}$ or $\rho < \bar{\rho}$.

We have run two sets of simulations, with dimension $d = 2$ and $d = 3$ respectively. In the case $d = 2$, we have repeated the previous procedure to construct 100 graphs for each $N \in \{10, 20, \ldots, 1000\}$, while in the case $d = 3$ we have constructed 50 graphs for each $N \in \{10, 20, \ldots, 250\}$. In Table 4.1 we report the values of the parameters c, p_e and of the geometric parameters of the family, used for both $d = 2$ and $d = 3$.

Once the graph is built, a reversible consensus matrix is produced according to the uniform weights protocol (3.11).

c	$\bar{s}$	$\bar{r}$	$\bar{\gamma}$	$\bar{\rho}$	p_e
0.5	0.1	1	0.9	0.052	0.8

TABLE 4.1
Parameters used in the simulations.

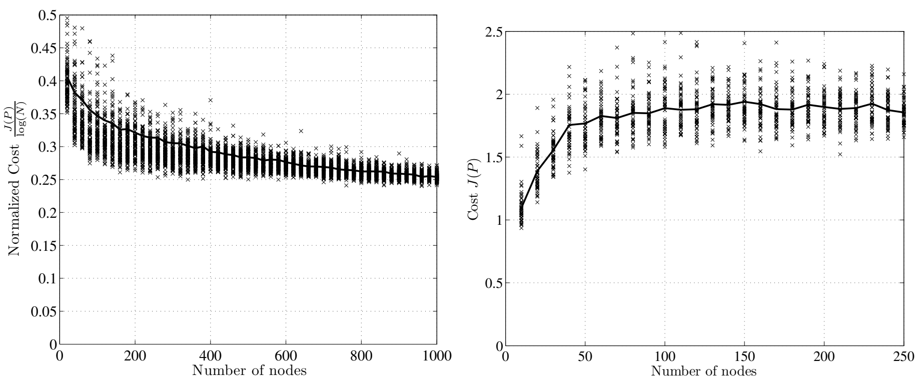

FIGURE 4.2. *Left panel: $d = 2$, $J(P)/\log N$ (100 realizations, average cost in solid line). Right panel: $d = 3$, $J(P)$ (50 realizations, average cost in solid line).*

The results are depicted in Figure 4.2. In the case $d = 2$, the cost $J(P_N)$ has been divided by $\log N$ which is its growth w.r.t. N predicted by Theorem 3.2.2. The experiments confirm the theory since the $J(P)$ appears to be confined in the the predicted bounds. The 2-dimensional and 3-dimensional case differ in the behavior for small values of N. Indeed, in the case $d = 2$ it seems from the experiments that, besides the dominant term scaling as $\log N$, there exists a decreasing higher order behavior. On the contrary, in the case $d = 3$ it appears that there is an increasing initial behavior.

5. The relation between the LQ cost and effective resistance: Proofs of Theorem 3.2.1 and Theorem 3.2.2. This section is devoted to the proofs of the theorems relating the LQ cost with the average effective resistance. We recall some useful facts from the literature, and then use these notions to prove the results.

5.1. Electrical networks: properties of the effective resistances. This section is devoted to briefly recall without proofs some well-known results on the behavior of the effective resistances in case of perturbation of the electrical network. These are of fundamental importance, since effective resistances show monotonicity properties which are not trivial to prove for consensus matrices without the electrical analogy.

A first important property is the following (see e.g. [27, Thm. B] for a proof).

LEMMA 5.1.1. *If the electrical network is connected, then the effective resistance is a distance. Namely, it satisfies the following properties:*

- $\mathcal{R}_{uv} \geq 0$ *for all* $u, v \in V$, *and* $\mathcal{R}_{uv} = 0$ *if and only if* $u = v$;
- $\mathcal{R}_{uv} = \mathcal{R}_{vu}$;
- $\mathcal{R}_{uw} \leq \mathcal{R}_{uv} + \mathcal{R}_{vw}$ *for all* $u, v, w \in V$.

A second result, known as Rayleigh's monotonicity law, says that increasing (resp., decreasing) the conductance in any edge of the network implies that the effective resistance between any other couple of nodes respectively cannot increase (resp., decrease).

The statement is essentially taken from [22, 11], where the authors were considering a more general case.

LEMMA 5.1.2 (**Rayleigh's monotonicity law**).
Let C and C' be the conductance matrices of two electrical networks such that

$$C_{uu'} \leq C'_{uu'},\, \forall\,(u,\,u') \in V \times V\,.$$

Then, the effective resistances between any two nodes v, v' in the network are such that

$$\mathcal{R}_{vv'}(C) \geq \mathcal{R}_{vv'}(C').$$

The following lemma [11, Lemma 4.6.1] says that, if we take two resistive networks with the conductance matrices scaled by a constant α, then the effective resistances will be scaled by the constant $1/\alpha$.

LEMMA 5.1.3.

$$\mathcal{R}_{uu'}(\alpha C) = \frac{1}{\alpha}\mathcal{R}_{uu'}(C),\, \forall\,(u,\,u') \in V \times V.$$

REMARK 5.1.1. *Lemma 5.1.3 and Rayleigh's monotonicity law imply that the effective resistance in an electrical network is essentially due to the graph topology. In fact, if we have an electrical network with conductance matrix C whose non-zero entries belong to the interval $[c_{\min},\, c_{\max}]$ and if C' is a conductance matrix having entries equal to 1 in the positions in which C has non-zero entries and to 0 elsewhere, then*

$$\frac{1}{c_{\max}}\mathcal{R}_{uu'}(C') \leq \mathcal{R}_{uu'}(C) \leq \frac{1}{c_{\min}}\mathcal{R}_{uu'}(C'),\, \forall\,(u,\,u') \in V \times V.$$

The last technical lemma deals with h-fuzzing in electrical networks with unitary conductances. Given an integer $h \geq 1$ and a graph $\mathcal{G}$, we call h-fuzz of $\mathcal{G}$, denoted by the symbol $\mathcal{G}^{(h)} = (V^{(h)}, \mathcal{E}^{(h)})$, a graph with the same set of nodes, $V^{(h)} = V$, and with an edge connecting two nodes u and v if and only if the graphical distance $d_{\mathcal{G}}(u,\,v)$ between u and v in $\mathcal{G}$ is at most h, namely

$$\mathcal{E}^{(h)} = \{(u,\,v) \in V \times V :\, d_{\mathcal{G}}(u,\,v) \leq h\}.$$

Notice that, if $h = 1$, then $\mathcal{G}^{(1)} = \mathcal{G}$. If D is the diameter of the graph, namely the maximum graphical distance between a couple of nodes, then $\mathcal{G}^{(D)}$ is the complete graph. It is easy to see that, if P is a stochastic matrix with positive diagonal entries, then the graph $\mathcal{G}_{P^h}$ associated with a P^h is the h-fuzz of the graph $\mathcal{G}_P$ associated with P.

The following lemma, which is stated with proof in [11, Lemma 5.5.1], suggests that the effective resistance of $\mathcal{G}$ and of its h-fuzz $\mathcal{G}^{(h)}$ have effective resistances with a similar asymptotic behavior.

LEMMA 5.1.4. *Let $h \in \mathbb{Z}$, $h \geq 1$, and let $\mathcal{G} = (V, \mathcal{E})$ be a graph and $\mathcal{G}^{(h)} = (V,\, \mathcal{E}^{(h)})$ be its h-fuzz. For any edge $e \in \mathcal{E}$, define $\mu_h(e)$ to be the number of paths of length at most h passing through e in $\mathcal{G}$ (without any self-loop in the path), and define $\mu_h = \max_{e\in\mathcal{E}} \mu_h(e)$. The following bounds hold true*

$$\frac{1}{h\mu_h}\mathcal{R}_{uv}(\mathcal{G}) \leq \mathcal{R}_{uv}(\mathcal{G}^{(h)}) \leq \mathcal{R}_{uv}(\mathcal{G}).$$

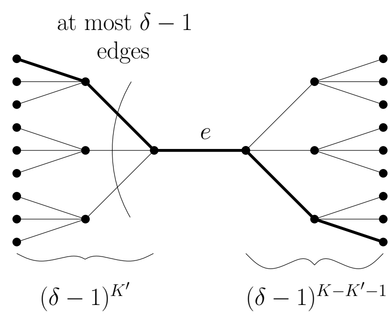

FIGURE 5.1. *Illustration of the proof of Lemma 5.1.5: upper bound on the number of paths of length K in which e is the $(K'+1)$-th edge, in a graph with node degree at most $\delta = 4$.*

The value of μ_h in the previous result clearly depends on the particular graph under consideration. The following lemma gives a conservative bound for μ_h which depends on the maximum degree of the nodes in the graph only.

LEMMA 5.1.5. *Let μ_h be defined as in Lemma 5.1.4. If in $\mathcal{G}$ all nodes have degree at most δ, then*

$$\mu_h \leq h^2(\delta-1)^{h-1}.$$

Proof. For any $K = 1, \ldots, h$, we want to find an upper bound on the number of paths of length K passing through the edge e. We let K' be an integer $0 \leq K' \leq K-1$, and we consider the number of paths in which edge e is the $(K'+1)$-th edge in the path, namely there are K' edges before e and $K - K' - 1$ edges after e. As it can be easily seen in Figure 5.1, there are at most $(\delta-1)^{K'}$ choices for portion of path preceding e, and at most $(\delta-1)^{K-K'-1}$ choices for the portion following e, so that there are at most $(\delta-1)^{K-1}$ paths having e in $(K'+1)$-th position. Summing upon all $K' = 0, \ldots, K-1$, and then summing also upon all path lengths $K = 1, \ldots, h$, we obtain that, for any $e \in \mathcal{E}$,

$$\mu_h(e) \leq \sum_{K=1}^{h} \sum_{K'=0}^{K-1} (\delta-1)^{K-1} = \sum_{K=1}^{h} K(\delta-1)^{K-1} \leq \sum_{K=1}^{h} h(\delta-1)^{h-1} = h^2(\delta-1)^{h-1}.$$

Finally notice that this upper bound for $\mu_h(e)$ is the same for all edges $e \in \mathcal{E}$, and thus it is also an upper bound for the maximum, μ_h. □

5.2. Proof of Theorem 3.2.1 and Theorem 3.2.2. The previous definitions and properties are used in this section to prove the main results Theorem 3.2.1 and Theorem 3.2.2 for reversible consensus matrices. Then Corollary 3.2.1 and Corollary 3.2.2 are immediate, since for symmetric matrices $\pi_i = \frac{1}{N}$ for all $i = 1, \ldots, N$. In order to prove the results, we need to introduce two more technical objects which will help us to develop the theory.

Consider a reversible consensus matrix P with invariant measure $\boldsymbol{\pi}$. We call *weighted cost* the following function of P

$$J_{\mathrm{w}}(P) := \operatorname{trace}\left[\sum_{t\geq 0}(I - \boldsymbol{\pi}\mathbf{1}^T)(P^T)^t \Pi P^t (I - \mathbf{1}\boldsymbol{\pi}^T)\right] \tag{5.1}$$

where $\Pi = \mathrm{diag}(\boldsymbol{\pi})$. Notice that in the case of symmetric matrices $J(P) = J_{\mathrm{w}}(P)$. Now, let $C := \Phi(P^2) = N\Pi P^2$. The second object we need is the *weighted average effective resistance*, which is defined as

$$\bar{\mathcal{R}}_{\mathrm{w}}(C) := \frac{1}{2}\boldsymbol{\pi}^T\mathcal{R}(C)\boldsymbol{\pi} = \frac{1}{2}\sum_{(u,\,v)\in V\times V}\mathcal{R}_{uv}(C)\pi_u\pi_v\,. \tag{5.2}$$

Again, notice that in the symmetric case the weighted definition coincides with the un-weighted one, namely $\bar{\mathcal{R}}(C) = \bar{\mathcal{R}}_{\mathrm{w}}(C)$. We present now a lemma which clarifies the relation between the costs $J(P)$ and $J_{\mathrm{w}}(P)$, and between the weighted and the un-weighted average effective resistances, respectively. The proof is immediate from the fact that $\pi_u > 0$ for all u.

LEMMA 5.2.1. *Let P be a consensus matrix with invariant measure $\boldsymbol{\pi}$ and let $C := \Phi(P^2) = N\Pi P^2$. Then*

$$\frac{1}{N\pi_{\max}}J_{\mathrm{w}}(P) \le J(P) \le \frac{1}{N\pi_{\min}}J_{\mathrm{w}}(P)$$

$$\pi_{\min}^2 N^2\bar{\mathcal{R}}(C) \le \bar{\mathcal{R}}_{\mathrm{w}}(C) \le \pi_{\max}^2 N^2\bar{\mathcal{R}}(C)\ .$$

The inequalities of the above lemma concern the LQ cost and the average effective resistance separately. Now, our goal is to find the relation between the cost of the consensus matrix P and the average effective resistance of the connected electrical network associated with P^2. Before doing so, we need the following technical lemma.

LEMMA 5.2.2. *If P is a consensus matrix, then the diagonal entries of its Green matrix G are positive.*

Proof. For ease of notation, we prove that $G_{11} > 0$; the proof for the other diagonal entries of G can be obtained by the same arguments.

We fix the following notation: we let $\boldsymbol{g}^T = [G_{11}, \tilde{\boldsymbol{g}}^T]$ be the first row of G, and we define the following partitions

$$L = \begin{bmatrix} l_{11} & \boldsymbol{r}_1^T \\ \boldsymbol{c}_1 & \tilde{L}\end{bmatrix},\ P = \begin{bmatrix} p_{11} & {\boldsymbol{r}'_1}^T \\ \boldsymbol{c}'_1 & \tilde{P}\end{bmatrix},\ \boldsymbol{g}^T = \begin{bmatrix}G_{11},\, \tilde{\boldsymbol{g}}^T\end{bmatrix},\ \boldsymbol{\pi}^T = \begin{bmatrix}\pi_1,\, \tilde{\boldsymbol{\pi}}^T\end{bmatrix}.$$

Because $GL = I - P$, we have $\boldsymbol{g}^T L = \boldsymbol{e}_1^T - \boldsymbol{\pi}^T$, where $\boldsymbol{e}_1$ denotes the first vector of the canonical basis of $\mathcal{R}^N$. Notice that $G\mathbf{1} = \mathbf{0}$ implies that $\boldsymbol{g}^T\mathbf{1} = 0$ and thus, in particular, $G_{11} = -\tilde{\boldsymbol{g}}^T\mathbf{1}_{N-1}$. Similarly, $\boldsymbol{\pi}^T L = \mathbf{0}^T$ gives $l_{11} = -\frac{1}{\pi_1}\tilde{\boldsymbol{\pi}}^T\boldsymbol{c}_1$. Hence, we can write the equality $\boldsymbol{g}^T L = \boldsymbol{e}_1^T - \boldsymbol{\pi}^T$ in the following equivalent way

$$\tilde{\boldsymbol{g}}_1^T\begin{bmatrix}-\mathbf{1}_{N-1} & I_{N-1}\end{bmatrix}\begin{bmatrix}-\frac{1}{\pi_1}\tilde{\boldsymbol{\pi}}^T \\ I_{N-1}\end{bmatrix}\tilde{L}\begin{bmatrix}-\mathbf{1}_{N-1} & I_{N-1}\end{bmatrix} = \boldsymbol{e}_1^T - \boldsymbol{\pi}^T.$$

By right-multiplying both sides of the above equality by the factor $\begin{bmatrix}\mathbf{0}_{N-1}^T \\ \tilde{L}^{-1}\end{bmatrix}\mathbf{1}_{N-1}$, we obtain that $\tilde{\boldsymbol{g}}_1^T\mathbf{1}_{N-1} = -\tilde{\boldsymbol{\pi}}^T\tilde{L}^{-1}\mathbf{1}_{N-1}$, and thus

$$G_{11} = \tilde{\boldsymbol{\pi}}^T\tilde{L}^{-1}\mathbf{1}_{N-1}\,.$$

Now notice that, by the definition $L = I - P$ of the Laplacian, $\tilde{L} = I_{N-1} - \tilde{P}$. Moreover, our definition of consensus matrix implies that P is primitive, and thus it is well-known that $\tilde{P}$ has all eigenvalues inside the unit circle (see e.g. [28, Lemma III.1] for a proof). This implies that $\tilde{L}$ is invertible, and that the series $\sum_{t\geq 0}\tilde{P}^t$ is convergent and is equal to $(I_{N-1}-\tilde{P})^{-1}=\tilde{L}^{-1}$. This allows to obtain

$$G_{11}=\tilde{\boldsymbol{\pi}}^T\sum_{t\geq 0}\tilde{P}^t\mathbf{1}_{N-1}\,.$$

Recalling that the entries of $\tilde{\boldsymbol{\pi}}$ are all positive, and that $\tilde{P}$ has non-negative entries with at least some positive element, this proves that $G_{11}>0$. □

Now we have the tools to prove the following lemma, which shows the relation between the weighted cost and the weighted average effective resistance.

LEMMA 5.2.3. *Let P be a reversible consensus matrix with invariant measure $\boldsymbol{\pi}$ and let $C:=\Phi(P^2)=N\Pi P^2$. Then*

$$\pi_{\min}N\bar{\mathcal{R}}_{\mathrm{w}}(C)\leq J_{\mathrm{w}}(P)\leq \pi_{\max}N\bar{\mathcal{R}}_{\mathrm{w}}(C).$$

Proof. To prove this lemma, we will prove the following two equalities involving the Green matrix associated with P^2, which we will denote by $G(P^2)$

1. $J_{\mathrm{w}}(P)=\operatorname{trace}\Pi G(P^2)$;
2. $\bar{\mathcal{R}}_{\mathrm{w}}(C)=\frac{1}{N}\operatorname{trace}G(P^2)$.

From such equalities, the statement follows, because Π is diagonal and positive definite, and $G(P^2)$ has positive diagonal entries (see Lemma 5.2.2).

As far as the first equality is concerned, observe that

$$\begin{aligned}J_{\mathrm{w}}(P)&=\operatorname{trace}\left(\sum_{t\geq 0}(P^t-\mathbf{1}\boldsymbol{\pi}^T)^T\Pi(P^t-\mathbf{1}\boldsymbol{\pi}^T)\right)\\&=\operatorname{trace}\left(\sum_{t\geq 0}\left((P^t)^T\Pi P^t-\boldsymbol{\pi}\boldsymbol{\pi}^T\right)\right)\\&=\operatorname{trace}\left(\sum_{t\geq 0}\Pi\left(P^{2t}-\mathbf{1}\boldsymbol{\pi}^T\right)\right)=\operatorname{trace}\left(\Pi G(P^2)\right).\end{aligned}$$

As far as the second equality is concerned, observe that, by substituting the expression for $\mathcal{R}_{uv}(C)$ given in Eq. (3.8) inside the definition of $\bar{\mathcal{R}}_{\mathrm{w}}(C)$, we get

$$\bar{\mathcal{R}}_{\mathrm{w}}(C)=\frac{1}{2}\sum_{u,v}\tfrac{1}{N}(\boldsymbol{e}_u-\boldsymbol{e}_v)^TG(P^2)\Pi^{-1}(\boldsymbol{e}_u-\boldsymbol{e}_v)\pi_u\pi_v,$$

from which we can compute

$$
\begin{aligned}
\bar{\mathcal{R}}_{\mathrm{w}}(C) &= \frac{1}{2}\sum_{u,v} \tfrac{1}{N}(\boldsymbol{e}_u - \boldsymbol{e}_v)^T G(P^2)\Pi^{-1}(\boldsymbol{e}_u - \boldsymbol{e}_v)\pi_u\pi_v \\
&= \frac{1}{N}\frac{1}{2}\sum_{u,v}(\boldsymbol{e}_u^T - \boldsymbol{e}_v^T)G(P^2)(\pi_v\boldsymbol{e}_u - \pi_u\boldsymbol{e}_v) \\
&= \frac{1}{N}\Big(\frac{1}{2}\sum_{u,v}(\pi_v\boldsymbol{e}_u^T G(P^2)\boldsymbol{e}_u + \pi_u\boldsymbol{e}_v^T G(P^2)\boldsymbol{e}_v) \\
&\qquad - \frac{1}{2}\sum_{u,v}(\pi_v\boldsymbol{e}_v^T G(P^2)\boldsymbol{e}_u + \pi_u\boldsymbol{e}_u^T G(P^2)\boldsymbol{e}_v)\Big) \\
&= \frac{1}{N}\left(\mathrm{trace}(G(P^2)) - \boldsymbol{\pi}^T G(P^2)\mathbf{1}\right),
\end{aligned}
$$

which yields the proof of the equality since $\boldsymbol{\pi}^T G(P^2)\mathbf{1} = 0$. □

These lemmas can be easily used to infer the first main result, since Theorem 3.2.1's proof follows immediately from the inequalities in Lemma 5.2.1 together with those in Lemma 5.2.3.

In order to prove the second main result, we need a last technical lemma, which allows us to reduce the computation of the average effective resistances on the 2-fuzz of $\mathcal{G}$ to those on $\mathcal{G}$ only.

LEMMA 5.2.4. *Let P be a reversible consensus matrix with invariant measure $\boldsymbol{\pi}$ and with associated graph $\mathcal{G}$. Let $C := \Phi(P^2)$. Then*

$$
\frac{1}{8N\pi_{\max}\delta^2 p_{\max}^2}\bar{\mathcal{R}}(\mathcal{G}) \le \bar{\mathcal{R}}(C) \le \frac{1}{N\pi_{\min}p_{\min}^2}\bar{\mathcal{R}}(\mathcal{G}), \tag{5.3}
$$

where δ denotes the largest degree of the graph nodes in $\mathcal{G}$ and $p_{\min}$ and $p_{\max}$ are, respectively, the minimum and the maximum of the non-zero elements of P.

Proof. First of all notice that, for all u, v such that $C_{uv} \neq 0$ we have that

$$
C_{uv} = N\pi_u[P^2]_{uv} = N\pi_u \textstyle\sum_w P_{uw}P_{wv}.
$$

By definition of $p_{\min}$ and $p_{\max}$, and because there are at most $\delta + 1$ non-zero terms P_{uw} for any fixed u, this yields

$$
\forall C_{uv} \neq 0,\ N\pi_{\min}p_{\min}^2 \le C_{uv} \le N\pi_{\max}(\delta+1)p_{\max}^2. \tag{5.4}
$$

Notice that $[P^2]_{uv} = \sum_w P_{uw}P_{wv} > 0$ if and only if there exists w such that $P_{uw} > 0$ and $P_{wv} > 0$, namely if their graphical distance is at most 2, where "at most" holds since $P_{vv} > 0, \forall v \in V$. This implies that the graph associated with P^2 is $\mathcal{G}^{(2)}$, the 2-fuzz of the graph $\mathcal{G}$ associated with P. By Remark 5.1.1, $\frac{1}{c_{\max}}\bar{\mathcal{R}}(\mathcal{G}^{(2)}) \le \bar{\mathcal{R}}(C) \le \frac{1}{c_{\min}}\bar{\mathcal{R}}(\mathcal{G}^{(2)})$, where $c_{\min}$ and $c_{\max}$ are the minimum and maximum non-zero entries of C, respectively. This, together with Eq. (5.4), gives

$$
\frac{1}{N\pi_{\max}(\delta+1)p_{\max}^2}\bar{\mathcal{R}}(\mathcal{G}^{(2)}) \le \bar{\mathcal{R}}(C) \le \frac{1}{N\pi_{\min}p_{\min}^2}\bar{\mathcal{R}}(\mathcal{G}^{(2)}).
$$

Then we apply Lemmas 5.1.4 and 5.1.5, both with $h = 2$, and we obtain

$$
\frac{1}{8(\delta-1)N\pi_{\max}(\delta+1)p_{\max}^2}\bar{\mathcal{R}}(\mathcal{G}) \le \bar{\mathcal{R}}(C) \le \frac{1}{N\pi_{\min}p_{\min}^2}\bar{\mathcal{R}}(\mathcal{G}),
$$

which yields the claim, because $(\delta+1)(\delta-1) \leq \delta^2$. □

Now we can prove the second main result: Theorem 3.2.2 immediately follows from Theorem 3.2.1 and Lemma 5.2.4.

6. Application to geometric graphs: Proof of Theorem 4.2.1. In order to prove Theorem 4.2.1, we need two preliminary results. The first one is an immediate corollary of a theorem taken from [10], which states that the claimed asymptotic behavior of geometric graphs holds true at least in the case of regular grids. Recall that, by regular grid, we mean a d-dimensional geometric graph with $N = n^d$ nodes lying on the points $(i_1, \ldots, i_d)$, where $i_1, \ldots, i_d \in \{0, \ldots, n-1\}$ and in which there is an edge connecting two nodes u, v if and only if their distance is $d_{\mathrm{E}}(u, v) \leq 1$.

LEMMA 6.0.5. *Let $B_{\mathcal{L}}$ be the incidence matrix of a regular grid in dimension d with $N = n^d$ nodes. Let $P \in \mathbb{R}^{N\times N}$ be the consensus matrix defined as follows*

$$P = I - \frac{1}{2d+1} B_{\mathcal{L}}^T B_{\mathcal{L}}$$

whose associated graph is the regular grid. Then

$$c_l f_d(N) \leq \bar{\mathcal{R}}(\mathcal{L}) \leq c_u f_d(N)$$

where c_l and c_u depend on d only, and where $f_d(N)$ is defined in Eq. (4.5).

Proof. With the same assumptions, from [10], Proposition 1, we know that

$$c_l' f_d(N) \leq J(P) \leq c_u' f_d(N)\,,$$

where c_l' and c_u' depend on δ only. The result immediately follows from Corollary 3.2.2. □

The second result allows us to reduce the problem of computing the average effective resistance in the geometric graph to the simpler case of two suitable grids. First of all, we state the following three technical results.

LEMMA 6.0.6. *In an hypercube $H \subseteq Q$ with side length less than $\frac{s}{\sqrt{d}}$, there is at most one node $u \in \mathcal{V}$. In an hypercube $H' \subseteq Q$ with side length greater than 2γ, there is at least one node $u' \in \mathcal{V}$.*

Proof. If the side length of an hypercube is $\frac{s}{\sqrt{d}}$, then its diagonal has length s. If we had two nodes in the hypercube, their distance would be less than s, in contradiction with the definition of s. The second claim is proved noticing that an hypercube of side length 2γ includes a sphere of radius γ. If it did not contain any node, then we could find a sphere of radius larger than γ not containing any node, in contradiction with the definition of γ. □

As a corollary of the previous lemma we have the following result.

LEMMA 6.0.7. *Let H be a hypercube in Q with edge length ℓ_H and let N_H be the number of nodes in it. Then*

$$\left\lfloor \frac{\ell_H}{2\gamma} \right\rfloor < \sqrt[d]{N_H} < \left\lceil \frac{\sqrt{d}\ell_H}{s} \right\rceil .$$

Proof. The result follows from Lemma 6.0.6 simply counting how many disjoint hypercubes of side length $\frac{s}{\sqrt{d}}$ and 2γ we can find in an hypercube of side length ℓ_H. □

In particular for the whole graph we have the following corollary.

COROLLARY 6.0.1. *The number of nodes N of the graph is such that*

$$\ell\frac{1-\frac{2\gamma}{\ell}}{2\gamma} < \sqrt[d]{N} < \ell\frac{\sqrt{d}-\frac{s}{\ell}}{s}.$$

Notice that, in the case where ℓ is big with respect to γ and s, the previous corollary essentially implies that N is proportional to ℓ^d. The following lemma concerns geometric graphs and their embeddings in lattices.

LEMMA 6.0.8. *Let $\mathcal{G}=(V,\mathcal{E})$ be a geometric graph with parameters (s, r, γ, ρ) and with nodes in an hypercube $Q=[0,\ell]^d$ in which $\gamma<\frac{\ell}{4}$. Then there exist two lattices, $\mathcal{L}_1$ and $\mathcal{L}_2$ such that*

$$k_1+q_1\bar{\mathcal{R}}(\mathcal{L}_1)\leq\bar{\mathcal{R}}(\mathcal{G})\leq k_2+q_2\bar{\mathcal{R}}(\mathcal{L}_2), \tag{6.1}$$

where q_1, q_2, k_1 and k_2 depend on s, r, γ, ρ, and on d. Moreover, there exist four constants, c_1', c_1'', c_2', and c_2'', depending on the same set of parameters, such that, if N_1 and N_2 are respectively the number of nodes of $\mathcal{L}_1$ and $\mathcal{L}_2$, then

$$c_1'N_1\leq N\leq c_1''N_1 \qquad c_2'N_2\leq N\leq c_2''N_2. \tag{6.2}$$

Proof. The idea is to tessellate the hypercube Q in order to obtain a rough approximation of $\mathcal{G}$, and then compute the bound for the effective resistance. Let us consider the upper bound first. Define $n_1:=\lceil\frac{\ell}{2\gamma}\rceil-1$ and $\lambda:=\frac{\ell}{n_1}$ and (exactly) tessellate the hypercube Q with $N_1:=n_1^d$ hypercubes of side length λ as in Figure 6.1. Notice that the technical assumption $\gamma<\frac{\ell}{4}$ also implies $\gamma<\frac{\ell}{2}$, which in turn avoids the pathological case in which $n_1=0$. Using the properties of $\lceil\cdot\rceil$, it can be seen that

$$2\gamma<\lambda<4\gamma.$$

Notice that the assumption $\ell>4\gamma$ ensures that $N_1\geq 2^d$.

Notice that, by Lemma 6.0.6, in each of these hypercubes there is at least one node of the graph $\mathcal{G}$. On the other hand, by Lemma 6.0.7, we can argue that in each hypercube there are at most $\lceil\frac{\sqrt{d}\lambda}{s}\rceil^d$ nodes. Since $2\gamma<\lambda$, then

$$\frac{1}{\lceil\frac{2\sqrt{d}\gamma}{s}\rceil^d}N\leq N_1\leq N.$$

This proves the first of the two bounds in Eq. (6.2). Another consequence of Lemma 6.0.6 is that for each hypercube we can select one "representative" node in V belonging to it. Let $V_{\mathcal{L}_1}\subseteq V$ be the set of these representatives. Now consider the regular lattice $\mathcal{L}_1=(V_{\mathcal{L}_1},\mathcal{E}_{\mathcal{L}_1})$, in which the set of nodes is $V_{\mathcal{L}_1}$, and in which the set of edges is built as follows: take two nodes u' and v' in $V_{\mathcal{L}_1}$, and consider the hypercubes which they represent, then construct the edge (u',v') if these two hypercubes touch each other (not diagonally). Define the function $\eta: V\to V_{\mathcal{L}_1}$ such that $\eta(u)=u'$ if u belongs to the hypercube associated with u'.

The next step is to prove that there exists an integer $h\geq 1$ such that the h-fuzz $\mathcal{G}^{(h)}$ of $\mathcal{G}$ embeds $\mathcal{L}_1$, namely that all the nodes and edges of $\mathcal{L}_1$ are also nodes and edges of $\mathcal{G}^{(h)}$. Take thus $u',v'\in V_{\mathcal{L}_1}$ such that $(u',v')\in\mathcal{E}_{\mathcal{L}_1}$. Their Euclidean distance is bounded as follows

$$d_{\mathrm{E}}(u',v')\leq\lambda\sqrt{d+3}$$

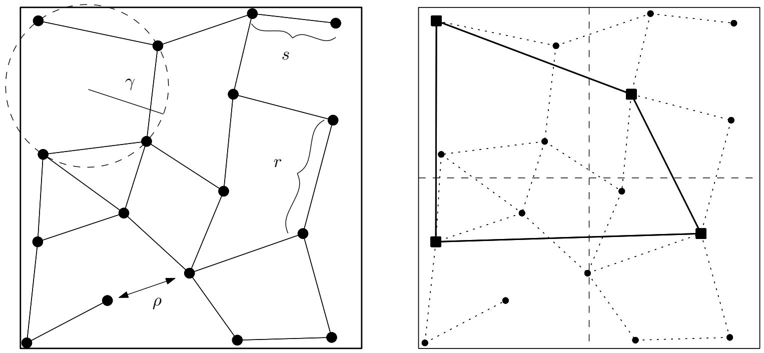

FIGURE 6.1. *On the left, an example of geometric graph in $\mathbb{R}^2$ with parameters s, r, γ and ρ (for ρ, the two nodes for which the minimum in the definition is attained). On the right, the lattice $\mathcal{L}_1$ built for the upper bound. The box-marked nodes are the representatives of the hypercubes, in thick solid line the edges of the lattice $\mathcal{L}_1$. Small nodes and dotted lines are the other nodes and edge of the original graph $\mathcal{G}$.*

as a simple geometric argument shows. By definition of ρ, we obtain

$$d_{\mathcal{G}}(u',\, v') \leq \frac{\lambda\sqrt{d+3}}{\rho} \leq \frac{4\gamma\sqrt{d+3}}{\rho}.$$

Take thus $h = \lfloor \frac{4\gamma\sqrt{d+3}}{\rho} \rfloor$ and build $\mathcal{G}^{(h)}$. By the previous discussion, it is manifest that $\mathcal{G}^{(h)}$ embeds $\mathcal{L}_1$.

Now, we claim that in $\mathcal{G}^{(h)}$ all the nodes lying in the same hypercube have graphical distance 1, namely they are all connected each other. In fact, if u and v lie in one hypercube, and thus $d_{\mathrm{E}}(u,\, v) \leq \lambda\sqrt{d}$, then we have

$$d_{\mathcal{G}}(v,\, u) \leq \frac{1}{\rho} d_{\mathrm{E}}(v,\, u) \leq \frac{\lambda\sqrt{d}}{\rho} \leq \frac{4\gamma\sqrt{d}}{\rho} \leq \frac{4\gamma\sqrt{d+3}}{\rho},$$

and thus $d_{\mathcal{G}}(v,\, u) \leq h$. This clearly yields

$$d_{\mathcal{G}^{(h)}}(u,\, \eta(u)) \leq 1. \tag{6.3}$$

We can now prove the claim. Since $\mathcal{G}^{(h)}$ embeds $\mathcal{L}_1$, by the properties of the effective resistances, for each $u', v' \in V_{\mathcal{L}_1}$ we have that

$$\mathcal{R}_{u'v'}(\mathcal{G}^{(h)}) \leq \mathcal{R}_{u'v'}(\mathcal{L}_1).$$

This is still limited to the set of representatives $V_{\mathcal{L}_1}$. If u and v are two generic nodes of $\mathcal{G}^{(h)}$, using Eq. (6.3) and the fact that the effective resistance is a distance (Lemma 5.1.1), we can obtain that

$$\begin{aligned}\mathcal{R}_{u,\,v}(\mathcal{G}^{(h)}) &\leq \mathcal{R}_{u,\,\eta(u)}(\mathcal{G}^{(h)}) + \mathcal{R}_{\eta(u),\,\eta(v)}(\mathcal{G}^{(h)}) + \mathcal{R}_{\eta(v),\,v}(\mathcal{G}^{(h)}) \\ &\leq 2 + \mathcal{R}_{\eta(u),\,\eta(v)}(\mathcal{G}^{(h)}).\end{aligned}$$

Thus, we have

$$\begin{aligned}\bar{\mathcal{R}}(\mathcal{G}^{(h)}) &= \frac{1}{2N^2}\sum_{u,v\in V}\mathcal{R}_{u,\,v}(\mathcal{G}^{(h)}) \le 1+\frac{1}{2N^2}\sum_{u,v\in V}\mathcal{R}_{\eta(u),\,\eta(v)}(\mathcal{G}^{(h)})\\ &\le 1+\frac{1}{2N^2}\sum_{u,v\in V}\mathcal{R}_{\eta(u),\,\eta(v)}(\mathcal{L}_1) = 1+\frac{1}{2N^2}\sum_{u',v'\in V_{\mathcal{L}_1}}\sum_{\substack{u\in\eta^{-1}(u')\\ v\in\eta^{-1}(v')}}\mathcal{R}_{u',v'}(\mathcal{L}_1)\\ &\le 1+\frac{M^2}{2N^2}\sum_{u',v'\in V_{\mathcal{L}_1}}\mathcal{R}_{u',v'}(\mathcal{L}_1) = 1+M^2\frac{N_1^2}{N^2}\bar{\mathcal{R}}(\mathcal{L}_1)\end{aligned}$$

where, as already pointed out, M, the maximum number of nodes of $\mathcal{G}$ in each hypercube of length λ, can be bounded as $M \le \lceil \frac{\sqrt{d}\lambda}{s} \rceil^d$. By previous arguments, $M\frac{N_1}{N}$ can be bounded from above by a constant dependent on the geometric parameters of the geometric graph and on d. Thus, the claim of the Lemma immediately descends from Lemma 5.1.4.

The proof for the lower bound follows basically the same steps once a good regular lattice candidate is selected. We tessellate again Q by means of hypercubes of side length

$$\lambda := \frac{\ell}{\lfloor \frac{\ell\sqrt{d}}{s} \rfloor + 1}$$

as in Figure 6.2. Observe that $\lambda < s/\sqrt{d}$ so that by Lemma 6.0.6 in each of them there can be at most one node. The candidate lattice is $\mathcal{L}_2 = (V_{\mathcal{L}_2}, \mathcal{E}_{\mathcal{L}_2})$, where $V_{\mathcal{L}_2}$ is the set of hypercubes and the edges connect again two nodes in $V_{\mathcal{L}_2}$ if the two corresponding hypercubes touch each other (not diagonally).

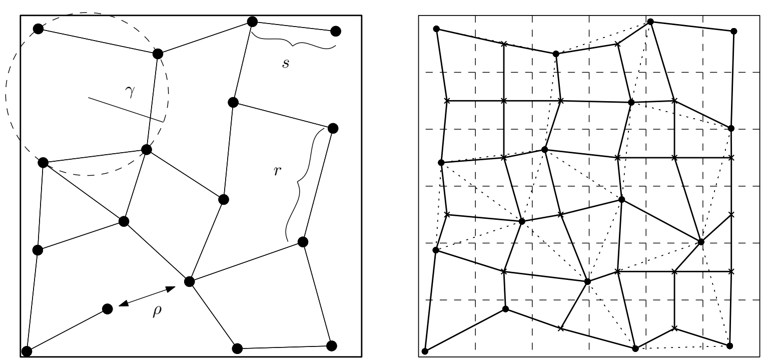

FIGURE 6.2. *On the left, the geometric graph already used for the upper bound. On the right, the lattice $\mathcal{L}_2$ built for the lower bound. The centers of the hypercubes in which there are no nodes of $\mathcal{G}$ are marked by a cross, while the bullet nodes are the nodes of $\mathcal{G}$. In solid lines are all the edges of $\mathcal{L}_2$, in dotted lines the other edges of the original graph $\mathcal{G}$.*

It can be proved that, if we take $u, v \in V$ such that $(u, v) \in \mathcal{E}$, then $d_{\mathcal{L}_2}(u, v) \le d\lceil r/\lambda \rceil$. We define now the map $\eta : V_{\mathcal{L}_2} \to V$ so that $\eta(u')$ is the node in V which is closest to u' in the Euclidean distance. It can be proved that, for all $u' \in V_{\mathcal{L}_2}$ we have that $d_{\mathrm{E}}(u', \eta(u')) \le \gamma$ and so for any pair of nodes u' and v' such that $\eta(u') = \eta(v')$ we have that $d_{\mathrm{E}}(u', v') \le 2\gamma$ and consequently $d_{\mathcal{L}_2}(u', v') \le d\lceil 2\gamma/\lambda \rceil$.

Analogously to the upper bound case, we write $V \subseteq V_{\mathcal{L}_2}$ identifying a node of the graph with the hypercube it belongs to. Once this is done, and taking $h := \max\{d\lceil r/\lambda\rceil, d\lceil 2\gamma/\lambda\rceil\}$ we can argue that $\mathcal{L}_2^{(h')}$ embeds $\mathcal{G}$ and that, given $u \in V$, for any pair of nodes u' and v' in $\eta^{-1}(u)$ we have

$$d_{\mathcal{L}_2^{(h)}}(u', v') \leq 1.$$

The last part of the proof, including the second bound in Eq. (6.2), is totally analogous to the upper bound case. □

We can now prove our theorem on geometric graphs.

Proof. [of Theorem 4.2.1] We know by Theorem 3.2.2 that

$$c_l \bar{\mathcal{R}}(\mathcal{G}) \leq J(P) \leq c_u \bar{\mathcal{R}}(\mathcal{G})$$

with c_l and c_u dependent on $p_{\min}$, $p_{\max}$, δ and the products $\pi_{\min}N$ and $\pi_{\max}N$. By Lemma 6.0.8, we can argue that

$$k_1' + q_1' \bar{\mathcal{R}}(\mathcal{L}_1) \leq J(P) \leq k_2' + q_1' \bar{\mathcal{R}}(\mathcal{L}_2) \tag{6.4}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two lattices such that $c_1' N_1 \leq N \leq c_1'' N_1$, $c_2' N_2 \leq N \leq c_2'' N_2$ and where k_1', q_1', k_2' and q_2' is a set of constants dependent on $p_{\min}$, $p_{\max}$, δ, the products $\pi_{\min}N$ and $\pi_{\max}N$, d and the parameters of the geometric graph.

Take now the grid $\mathcal{L}_1$, let $B_{\mathcal{L}_1}$ be its adjacency matrix, and build the consensus matrix

$$P_1 = I - \frac{1}{2d+1} B_{\mathcal{L}_1}^T B_{\mathcal{L}_1}.$$

By Lemma 6.0.5, we know that

$$\alpha_1 f_d(N_1) \leq \bar{\mathcal{R}}(\mathcal{L}_1) \leq \alpha_2' f_d(N_1),$$

where α_1 and α_2 depend on the parameter δ only.

Notice now that Lemma 6.0.8 also states that $c_1' N_1 \leq N \leq c_1'' N_1$, where c_1' and c_1'' depend on the parameters of the geometric graph only. Simple computations using the definition of $f_d(\cdot)$ in Eq. (4.5) yield

$$k_1' + q_1' f_d(N) \leq \bar{\mathcal{R}}(\mathcal{L}_1) \leq k_1'' + q_1'' f_d(N),$$

where k_1', q_1', k_1'' and q_1'' depend on the geometric parameters and on d.

Analogously, there exists a symmetric consensus matrix P_2 associated with $\mathcal{L}_2$ for which

$$k_2' + q_2' f_d(N) \leq \bar{\mathcal{R}}(\mathcal{L}_1) \leq k_2'' + q_2'' f_d(N),$$

where k_2', q_2', k_2'' and q_2'' depend on the geometric parameters and on d.

By substituting in Eq. (6.4), it is now clear that

$$k_1 + q_1 f_d(N) \leq J(P) \leq k_2 + q_2 f_d(N)$$

with k_1, q_1, k_2 and q_2 as in the statement of the theorem. □

7. Conclusion. In this paper we study an LQ cost which measures the performance of a consensus algorithm. We show that, under mild assumptions on the associated communication graph, if the consensus matrix is reversible then the LQ cost can be bounded using a quantity related to the graph topology only, namely its average effective resistance. For the generic reversible matrix a strong condition on the entries of the invariant measure must be satisfied in order the bound to be effective. However, we have shown that some highly popular and easily implementable strategies implicitly fulfil it. Exploiting this result, we study a large class of graphs, called geometric graphs, which describe geometrically local communication and which can be seen as perturbed grids. We show the the cost exhibits a particular behaviour as a function of the number of nodes in the graph, which is related to the dimension of the space in which the graph is drawn only, extending a result already known for highly symmetric graphs.

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