A resistance-based approach to consensus algorithm performance analysis

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***Abstract*—**

We study the well known linear consensus algorithm by means of a LQ-type performance cost. We want to understand how the communication topology influences this algorithm. In order to do this, we recall the analogy between Markov Chains and electrical resistive networks. By exploiting this analogy, we are able to rewrite the performance cost as the average effective resistance on a suitable network. We use this result to show that if the communication graph fulfills some local properties, then its behavior can be approximated with that of a suitable grid, over which the behavior of the cost is known.

I. INTRODUCTION

The last two decades have witnessed a great effort spent by several scientific communities in the development and in the analysis of multi-agents systems. The large number of simple intercommunicating and interacting entities can be profitably used in order to model a number of rather different applications: just to recall some of them, coordinated control [1], distributed estimation [2], load balancing [3], sensor calibration for sensor networks [4].

One of the most studied distributed averaging algorithms is discrete-time linear consensus. In this protocol, we assume to have N simple agents each of them memorizing a real value $x_u(t)$, $u = 1, \ldots, N$ at time $t \geq 0$. Moreover, we assume there exists a graph $\mathcal{G} = (V, \mathcal{E})$, in which the set of nodes V can be put in bijective correspondence with the set of agents (so we can identify them), and in which there is a directed edge (u, v) if and only if the agent u can communicate its value x_u to agent v. We call this graph the communication graph.

The goal of linear consensus is to drive each agent to asymptotically reach the same value. Let $\mathcal{N}_u$ to be the set of neighbors of agent u (not containing u itself), and stack all the $x_u(t)$'s in a vector $\boldsymbol{x}(t) \in \mathbb{R}^N$. We let this vector evolve as follows

$$\boldsymbol{x}(t+1) = P\boldsymbol{x}(t) \tag{1}$$

where P is *consistent*[1] with the graph $\mathcal{G}$, and it is stochastic, namely with non-negative entries and such that $P\mathbf{1} = \mathbf{1}$ ($\mathbf{1}$ is a column vector with all entries equal to 1 and of suitable dimension), aperiodic and irreducible[2]. In words, each sensor receives the values from all its neighbors, and then updates its value with a convex combination of them and its previous one. It is well known that, under the given assumptions, we have

$$x_u(t) \stackrel{t\to\infty}{\longrightarrow} \alpha, \, \forall\, u = 1, \ldots, N$$

with $\alpha = \boldsymbol{\pi}^T \boldsymbol{x}(0)$, where $\boldsymbol{\pi}^T$ is the left invariant normalized eigenvalue of P, namely $\boldsymbol{\pi}^T P = \boldsymbol{\pi}^T$ and $\pi^T \mathbf{1} = 1$.

In this paper we put several constraints on P and $\mathcal{G}$. First of all, we assume $\mathcal{G}$ to be undirected. Concerning P, we shall assume it is symmetric. In this case, $\boldsymbol{\pi} = N^{-1}\mathbf{1}$ and the algorithm is called average linear consensus, because the asymptotic value is the average of the initial conditions.

As recent papers have underlined, the classical performance index of P, the convergence speed, is not the only index for the performance evaluation. Different costs arise from different problems, and it can be shown by examples that considering a different performance index can indeed lead to different optimal graph topologies. In this paper, we consider an LQ cost which is familiar to control theorists. We consider the case when the initial condition is a random variable with zero-mean and covariance matrix $\mathbb{E}\left[\boldsymbol{x}(0)\boldsymbol{x}(0)^T\right] = I$, and we study the expectation of the norm of the trajectory of the states:

$$J(P) := \frac{1}{N} \sum_{t \geq 0} \mathbb{E}\left[\|\boldsymbol{x}(t) - \boldsymbol{x}(\infty)\|^2\right]. \tag{2}$$

[1] We define the graph G_P *associated* with P as $\mathcal{G}_P = (V, \mathcal{E})$, with $(u, v) \in \mathcal{E}$ if and only if $P_{uv} \neq 0$. We say that P is consistent with a graph $\mathcal{G}$ is $\mathcal{G}_P$ is a subgraph of $\mathcal{G}$.

[2] P is aperiodic if the greatest common divisor of the lengths of all cycles in its associated graph $\mathcal{G}_P$ is one. E.g., the presence of a self-loop implies aperiodicity. P is irreducible if $\mathcal{G}_P$ is strongly connected, namely, for all $u, v \in V$, there exists a path connecting u to v.

The same cost arises also in quite different frameworks, e.g. from consensus algorithm in the presence of noise [5], or from a formation-control problem [6], and this relevance in such different scenarios makes it worth to be studied.

Finding $J(P)$ for a generical P is obviously just a matter of computation, so we would prefer to have other tools to estimate its value. The first goal of this paper is to unveil the role played by the graph topology by showing how the computation of $J(P)$ is equivalent to the computation of the average effective resistance in a suitable electric network. The second main goal is to use this result in order to estimate $J(P)$ for an entire family of graphs, which we call geometric graphs. Intuitively, these graphs are "perturbed grids", and in fact we will show in Sect. V that $J(P)$ in such a graph is, up to multiplicative constants, the same as in a grid.

II. Linear Algebra Preliminaries

In this section we are going to present some basic linear algebra facts which will be useful throughout the paper.

We call a matrix $L \in \mathbb{R}^{N\times N}$ a Laplacian matrix if

$$\begin{cases} L_{ij} < 0 & i \neq j \\ L\mathbf{1} = \mathbf{0} \end{cases} . \tag{3}$$

Under the assumption that the graph associated with L is strongly connected, it is well-known [7] that $\dim\ker L = 1$. This implies that also the left kernel of L has dimension 1: let $\boldsymbol{\pi}^T$ the unique vector such that $\boldsymbol{\pi}^T L = 0$ and $\boldsymbol{\pi}^T\mathbf{1} = 1$.

It is easy to see that the matrix

$$\bar{L} := \begin{bmatrix} L & \mathbf{1} \\ \boldsymbol{\pi}^T & 0 \end{bmatrix}$$

is invertible. To solve the problem of finding the inverse, we use the notion of Green matrix of the Laplacian L.

Definition 2.1: Let L be a Laplacian matrix such that $L\mathbf{1} = 0$ and $\boldsymbol{\pi}^T L = 0$, where $\boldsymbol{\pi}^T\mathbf{1} = 1$. Then, the Green matrix of L is the unique matrix X such that

$$\begin{cases} XL = I - \mathbf{1}\boldsymbol{\pi}^T \\ X\mathbf{1} = 0 \end{cases} . \tag{4}$$

The following proposition is just a matter of computation.

Proposition 2.1: The inverse of $\bar{L}$ is

$$\bar{L}^{-1} = \begin{bmatrix} X & \mathbf{1} \\ \boldsymbol{\pi}^T & 0 \end{bmatrix}, \tag{5}$$

where the matrix X is the Green matrix of L.

These notions are particularly useful in the case when L is the Laplacian matrix associated with a Markov chain. Let P be a stochastic matrix (and thus it can be interpreted as the transition probability matrix of a Markov chain), and define $L_P := I - P$. It is immediate to check that L_P is indeed a Laplacian matrix as defined above. It is also easy to verify that the Green matrix of L_P, which without confusion we will call the Green matrix of P, is given by the following expression.

Lemma 2.1: The Green matrix of P is

$$X = \sum_{t\geq 0} \left(P^t - \mathbf{1}\boldsymbol{\pi}^T\right), \tag{6}$$

where $\boldsymbol{\pi}$ is the left normalized eigenvector of P relative to the eigenvalue 1.

In the particular case when P is symmetric (and hence the same holds for L), we have the following expression for X:

$$X = \sum_{t\geq 0} \left(P^t - \frac{1}{N}\mathbf{1}\mathbf{1}^T\right). \tag{7}$$

In the next two sections we are going to show

- How X is related to the effective resistances of an electrical network defined in a suitable way.
- How X is related to the performance cost for the consensus algorithm we want to study.

III. Electrical analogy

In this section, we present an analogy between resistive electrical networks and reversible Markov chains, i.e., Markov chains such that the invariant measure $\boldsymbol{\pi}^T$ and the transition matrix P satisfy the relation

$$\pi_u P_{uv} = \pi_v P_{vu}\,\forall u, v\,.$$

This analogy, first noticed by Doyle and Snell in [8], allows simpler proofs for some properties of the Markov chain, and a physical intuition. Here, we will focus our attention on Markov chains with symmetric P, which are clearly a particular case of reversible chains.

We define a resistive electrical network as a pair $(\mathcal{G}, C)$, or equivalently $(\mathcal{G}, R)$, where:

- $\mathcal{G}$ is an undirected graph (without self-loops), with N vertices, and M edges;
- C and R are two functions associating to each edge of the graph a strictly positive number, called respectively the conductance and the resistance of the edge, one the inverse of the other.

Actually, we can consider any non-existing edge to be given zero conductance or infinite resistance.

We will use the notational convention to see $\mathcal{G}$ as a directed graph $\mathcal{G} = (V, \mathcal{E})$, where each undirected edge is replaced by two directed edges, one for each direction, so that $|\mathcal{E}| = 2M$. With this notation, the conductance is a function $C : \mathcal{E} \to [0, +\infty)$ such that $C((u,v)) = C((v,u))$ for any edge (u,v). We will use two functions $h : \mathcal{E} \to V$ and $t : \mathcal{E} \to V$ to denote the head (starting vertex) and the tail (ending vertex) of each directed edge, respectively. We define the incidence matrix $A \in \{0, \pm 1\}^{2M \times N}$ as follows

$$A_{eu} = \begin{cases} -1 & \text{if } u = t(e) \\ 1 & \text{if } u = h(e) \\ 0 & \text{otherwise} \end{cases}$$

Finally, we define the matrix $\mathcal{C} \in \mathbb{R}^{2M \times 2M}$ as a diagonal matrix, where the diagonal entries are the conductances, namely $\mathcal{C}_{ee} = C(e)$ for all $e \in \mathcal{E}$. It is now immediate to obtain

$$[A^T \mathcal{C} A]_{uv} = \begin{cases} 2c_u & \text{if } u = v \\ -2C(e) & \text{if } (u,\, v) = e \in \mathcal{E} \\ 0 & \text{if } (u,\, v) \notin \mathcal{E} \end{cases},$$

where $c_u := \sum_{e|u=t(e)} C(e)$ is the sum of all the conductances of the edges incoming in u.

From the matrices A and $\mathcal{C}$ is it possible to construct reversible Markov chains with any prescribed invariant measure $\boldsymbol{\pi}$. We will focus here on the construction of symmetric matrices, i.e., with $\boldsymbol{\pi} = \frac{1}{N}\mathbf{1}$. To this aim, simply define $L := \frac{1}{2} A^T \mathcal{C} A$, and notice that this is indeed a symmetric Laplacian; its associated Markov chain is clearly given by $P = I - L$. In order to assure P to have all non-negative components, we should write $P = I - \frac{1}{c} L$, with $c = \sum_{u,\, v} C_{(u,v)}$: for simplicity and without loss of generality, we consider $c = 1$. Vice-versa, from any reversible Markov chain it is possible to construct a corresponding electrical network (unique, up to normalization of the total sum of conductances in the network); for symmetric P, the construction is simply to re-use the same graph associated with P (except self-loops), and to let $C((u,v,)) = P_{uv}$ for any non-zero entry of P.

We define, as usual, the effective resistance between to nodes u and v in the electric network $(\mathcal{G}, C)$ the quantity

$$\mathcal{R}_{uv}(\mathcal{G},\, C) = \frac{v_u - v_v}{I},$$

where v_u and v_v are the potentials at nodes respectively u and v when we inject a current of value I in u and we extract the same from v. We are now going to show how to obtain the effective resistances between any two nodes from the Green matrix X of the Laplacian $L = \frac{1}{2} A^T \mathcal{C} A$, following a line which is very close to the arguments used in [9].

Under full generality, suppose $\boldsymbol{i} \in \mathbb{R}^N$ is such that $\boldsymbol{i}^T \mathbf{1} = 0$: the entries of $\boldsymbol{i}$ represent the current which is injected (or extracted if negative in sign) in each node of the network. For example, if we inject I ampere in node u and extract I ampere from node v, we have $\boldsymbol{i} = I(\boldsymbol{e}_v - \boldsymbol{e}_u)$, where $\boldsymbol{e}_k$ denotes the element of the canonical base of $\mathbb{R}^N$ having a one in position k.

Let now $j \in \mathbb{R}^M$ be a vector such that

$$A^T \boldsymbol{j} = \boldsymbol{i}: \tag{8}$$

we call $\boldsymbol{j}$ a flow through the network satisfying $\boldsymbol{i}$, in the sense that, if $\boldsymbol{j}$ is seen as an edge-valued function, it respects Kirchhoff's law in each node. Moreover, we say $\boldsymbol{j}$ is a current if there exists $\boldsymbol{v} \in \mathbb{R}^N$ such that

$$\mathcal{C} A \boldsymbol{v} = \boldsymbol{j} \tag{9}$$

and we call $\boldsymbol{v}$ the potential on the network. Using Eq. (8) and Eq. (9) we obtain

$$A^T \mathcal{C} A \boldsymbol{v} = \boldsymbol{i} \Longrightarrow L\boldsymbol{v} = \frac{1}{2}\boldsymbol{i}. \tag{10}$$

We say we can solve the electrical network if we are able to find out a suitable $\boldsymbol{v}$ and consequently a suitable $\boldsymbol{j}$ which respect Eq. (8) and Eq. (9). In order to obtain an unique solution for Eq. (10), let's consider the vector $\boldsymbol{\pi} = \frac{1}{N}\mathbf{1}$, and assume for the potential the following constraint

$$\boldsymbol{\pi}^T \boldsymbol{v} = 0. \tag{11}$$

Because $\boldsymbol{\pi}$ is not orthogonal to $\mathbf{1}$, this allows $\boldsymbol{v}$ to be uniquely determined. Observe now that, using Eq. (10) and Eq. (11), we obtain the following equation which solves the electric network:

$$\begin{bmatrix} L & \mathbf{1} \\ \frac{1}{N}\mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\boldsymbol{i} \\ 0 \end{bmatrix}. \tag{12}$$

Recall now from Sect. II that from Eq. (5) we immediately obtain

$$\boldsymbol{v} = X \frac{1}{2} \boldsymbol{i}.$$

Now, set $\boldsymbol{i} = \boldsymbol{e}_u - \boldsymbol{e}_v$, so that

$$v_u - v_v = \boldsymbol{i}^T \boldsymbol{v} = \tfrac{1}{2} \boldsymbol{i}^T X \boldsymbol{i}.$$

Because the value of injected current is $I = 1$, this yields to

$$\mathcal{R}_{uv}(\mathcal{G},\, R) = \tfrac{1}{2}(\boldsymbol{e}_u - \boldsymbol{e}_v)^T X (\boldsymbol{e}_u - \boldsymbol{e}_v). \tag{13}$$

A. The properties of effective resistance

As we will show in the next section, the effective resistances in the electric network defined above can be used in order to compute the performance cost for the consensus algorithm. This seems actually extremely appealing because effective resistances show both a monotonicity law and a substantial invariance for h-fuzzing and re-scaling, as stated in the following lemmas.

The following first theorem is called Rayleigh's monotonicity law, and it basically says that if in an electric network we add edges or we reduce the resistance in any edge, the effective resistance cannot increase, while if we cut edges or increase the resistance of any edge, the effective resistance cannot decrease. The statement is essentially taken from [10], where the authors referred to a more general case.

*Theorem 3.1 (**Rayleigh's monotonicity law**):* Let $(\mathcal{G}, R)$ and $(\tilde{\mathcal{G}}, \tilde{R})$ be two electric networks such that $\mathcal{G}$ has the same nodes and a subset of the edges of $\tilde{\mathcal{G}}$, and $R(e) \geq \tilde{R}(e)$ for every common edge of the two graphs.

Then, the effective resistances between any two nodes in the two networks are such that

$$\mathcal{R}_{uv}(\mathcal{G}, R) \geq \mathcal{R}_{uv}(\tilde{\mathcal{G}}, \tilde{R}). \tag{14}$$

Remark 3.1: It is intuitive and not difficult to show that in a network with all the resistances equal to r_0, the effective resistance between two generic nodes is exactly r_0 times the effective resistance among the same nodes, in the same graph but having set all resistances equal to 1. This, together with Rayleigh's monotonicity law, implies that the effective resistance in an effective network is essentially due to the graph topology. This justify the notation $\mathcal{R}_{u,v}(\mathcal{G})$ which we will use to denote the effective resistance on the graph $\mathcal{G}$, when all the resistances are set to 1 Omh, while we denote by $\mathcal{R}_{uv}(\mathcal{G}, \mathcal{C})$ the effective resistance between two nodes in a network with general conductances C. If C is such that all the resistances $1/C(e)$ belong to an interval $[r_{\min}, r_{\max}]$, then we will have that

$$r_{\min}\mathcal{R}(\mathcal{G})_{u,v} \leq \mathcal{R}_{uv}(\mathcal{G}, \mathcal{C}) \leq r_{\max}\mathcal{R}_{uv}(\mathcal{G})\,.$$

Let's restrict, now, to the case of unitary resistances. The following lemma deals with h-fuzzing. Given an integer $h \geq 1$, we say that a graph $\mathcal{G}^{(h)} = (V^{(h)}, \mathcal{E}^{(h)})$ is the h-fuzz of another graph $\mathcal{G} = (V, \mathcal{E})$ if $V^{(h)} = V$, namely they share the same set of nodes, and there is an edge from u to v in $\mathcal{G}^{(h)}$ as soon as the graphical distance[3] from u to v in $\mathcal{G}$ is less than or equal to h. The extreme cases are $h = 1$, for which the 1-fuzz is simply the original graph, and $h \geq D$, D diameter of the original graph, for which the h-fuzz is the complete graph.

Lemma 3.1: Let $h \in \mathbb{Z}$, $h \geq 1$, and $\mathcal{G} = (V, \mathcal{E})$ be a graph, and assume $\mathcal{G}^{(h)} = (V, \mathcal{E}^{(h)})$ to be its h-fuzz. Then, given $e \in \mathcal{E}$, define $\mu(e)$ as the number of paths of length at most h passing through e in $\mathcal{G}$, and $\mu = \max_e \mu(e)$. We have

$$\frac{1}{h\mu}\mathcal{R}_{uv}(\mathcal{G}) \leq \mathcal{R}_{uv}(\mathcal{G}^{(h)}) \leq \mathcal{R}_{uv}(\mathcal{G}).$$

Remark 3.2: Observe that if in the graph $\mathcal{G}$ each node has at most δ neighbors, then $\mu \leq \delta^h$.

IV. Performance of consensus algorithm in terms of effective resistances

In this section, we are going to show how to rewrite the LQ cost $J(P)$, which evaluates the performance of the consensus algorithm, in terms of effective resistances of a suitable electrical network.

We recall that, given the consensus protocol

$$\boldsymbol{x}(t+1) = P\boldsymbol{x}(t)$$

the performance measure we want to evaluate is

$$J(P) := \frac{1}{N}\sum_{t\geq 0}\mathbb{E}\left[\|\boldsymbol{x}(t) - \boldsymbol{x}(\infty)\mathbf{1}\|^2\right]\,,$$

where the expectation is with respect to the random initial condition. Under the assumption that the initial condition has zero-mean and covariance matrix $\mathbb{E}\left[\boldsymbol{x}(0)\boldsymbol{x}(0)^T\right] = I$, it is easy to show (see [11]) that, for symmetric P,

$$J(P) = \frac{1}{N}\operatorname{Tr}\sum_{t\geq 0}\left[P^{2t}(I - \mathbf{1}\mathbf{1}^T)\right]. \tag{15}$$

Now construct an electrical network associated with the matrix P^2 in the following way. As a graph, consider the graph associated with P^2; notice that, thanks to the assumption that P has all self-loops, this graph is exactly the 2-fuzz of the graph $\mathcal{G}_P$ associated with P. Then set the conductances to be $C\left((u,v)\right) = [P^2]_{u,v}$ for all edge (u, v).

Theorem 4.1: Given a stochastic, symmetric, aperiodic and irreducible matrix P, the associated LQ cost

[3]The graphical distance between two nodes in a graph is the length (number of edges) of the shortest path connecting them.

defined in Eq. 2 is equal to

$$J(P) = \overline{\mathcal{R}} := \frac{1}{N^2} \sum_{u \neq v} \mathcal{R}_{uv}, \tag{16}$$

where $\mathcal{R}_{uv}$ denotes the effective resistance between nodes u and v in the above-constructed electric network associated with P^2. Namely, $J(P)$ is the average of the effective resistances in the whole network.

Proof: Consider the Laplacian of the matrix P^2, $L = I - P^2$. Having built the electric network as in the statement of the theorem, it turns immediately out that, being A the incidence matrix of the graph underlying the network, we have $L = \frac{1}{2} A^T \mathcal{C} A$.

Recalling the definition of the Green matrix in Eq. (6), we easily observe that Eq. (15) gives

$$J(P) = \tfrac{1}{N} \operatorname{Tr} X$$

where X here denotes the Green matrix of P^2. Now we can conclude by exploiting the relationship between effective resistances and Green matrix

$$\sum_{u \neq v} \mathcal{R}_{uv} = \frac{1}{2} \sum_{u \neq v} (\boldsymbol{e}_u - \boldsymbol{e}_v)^T X (\boldsymbol{e}_u - \boldsymbol{e}_v) = N \operatorname{Tr} X$$

where we used the fact that $X\mathbf{1} = \mathbf{0}$. This immediately yields

$$J(P) = \frac{1}{N} \operatorname{Tr} X = \frac{1}{N^2} \sum_{u \neq v} \mathcal{R}_{uv},$$

which is the thesis. ∎

Remark 4.1: The previous result easily yields that under mild assumptions the cost $J(P)$ is mainly due to the topology of the graph $\mathcal{G}_P$ associated with P. In fact, let us denote by $(\mathcal{G}_{P^2}, C)$ the above-constructed electric network associated with P^2, and call $\overline{\mathcal{R}}(\mathcal{G}_{P^2}, C)$ and $\overline{\mathcal{R}}(\mathcal{G}_{P^2})$ the average effective resistances in the networks whose graph is $\mathcal{G}_{P^2}$) and with conductances respectively given by C and all set to 1. By Thm. 3.1 and Remark 3.1, we have

$$r_{\min} \overline{\mathcal{R}}(\mathcal{G}_{P^2}) \leq \overline{\mathcal{R}}(\mathcal{G}_{P^2}, C) \leq r_{\max} \overline{\mathcal{R}}(\mathcal{G}_{P^2}),$$

where $r_{\min}$ and $r_{\max}$ are respectively the smallest and the largest resistances in $(\mathcal{G}_{P^2}, C)$. Now we recall that $\mathcal{G}_{P^2}$ is the 2-fuzz of $\mathcal{G}_P$, and so, using Lemma V-B, we have

$$\frac{r_{\min}}{2\mu} \overline{\mathcal{R}}(\mathcal{G}_P) \leq \overline{\mathcal{R}}(\mathcal{G}_{P^2}, C) \leq r_{\max} \overline{\mathcal{R}}(\mathcal{G}_P),$$

so the middle term, which by Theorem 4.1 is equal to $J(P)$, depends on the topology of $\mathcal{G}_P$ only, up to constants. For this reason, given a graph $\mathcal{G}$, we will from now on often refer to the cost of $\mathcal{G}$, meaning the "characteristic behavior" of the cost for any P associated with $\mathcal{G}$. Observe that the bound depends on μ and hence, from 3.2, on the number of neighbors, which, for large scale graphs, should remain bounded. This assumption is rather mild, and it is satisfied in our application at Sect. V

V. Exploiting Lattices

In this section we are going to apply the previous results in order to show how to estimate the cost $J(P)$ in the case its associated graph $\mathcal{G}_P$ is what we call a geometric graph. Here we refer to geometric graphs as graphs which can be thought to have a "real" counterpart, deployed in an hypercube in $\mathbb{R}^d$, as explained below. The goal is to show that, for what attains to the performance cost (see Sect. IV), $\mathcal{G}$ can be approximated with two suitable grids, whose dimensions depend only on some geometric parameters characterizing $\mathcal{G}_P$.

A. Geometric graphs: definitions and main result

Let's consider a connected undirected graph $\mathcal{G} = (V, \mathcal{E})$ such that $V \subset \mathbb{R}^d$ and $|V| = N$. Let's moreover assume that $Q \subset \mathbb{R}^d$ is an hypercube of edge length ℓ in $\mathbb{R}^d$ such that $V \subset Q$, namely such that Q contains every node of the graph.

Following [10], we define over $\mathcal{G}$ and Q the following parameters:

- the minimum Euclidean distance[4] between any two nodes:
$$s = \inf_{u,\, v \in V,\, u \neq v} \{d_{\mathrm{E}}(u, v)\}\,; \tag{17}$$
- the maximum Euclidean distance between any two connected nodes:
$$r = \sup_{(u,\, v) \in \mathcal{E}} \{d_{\mathrm{E}}(u, v)\}\,; \tag{18}$$
- the radius, γ, of the largest ball centered in Q not containing any node of the graph:
$$\gamma = \max\{r |\, B(x, r) \cap V = \emptyset, \forall\, x \in Q\}\,; \tag{19}$$
- the minimum ratio between the Euclidean distance of two nodes and their graphical distance,
$$\rho = \min\left\{ \frac{d_{\mathrm{E}}(u, v)}{d_{\mathcal{G}}(u, v)} |\, (u, v) \in V \times V \right\}. \tag{20}$$

The main result of this section is the following.

[4]Given a generic graph $\mathcal{G} = (G, \mathcal{E})$ and two nodes $u, v \in G$ deployed in $\mathbb{R}^d$, we will denote with $d_{\mathrm{E}}(u, v)$ the Euclidean distance between u and v (in $\mathbb{R}^d$), and with $d_{\mathcal{G}}(u, v)$ their graphical distance (in $\mathcal{G}$).

Theorem 5.1: Assume P to be a stochastic, symmetric, aperiodic and irreducible matrix, associated with a geometric graph $\mathcal{G}_P = (V, \mathcal{E})$, and assume all the nonzero entries of P lie in an interval $[p_{\min}, p_{\max}]$. Moreover, assume that $V \subset Q \subset \mathbb{R}^d$. Then there exist two lattices, $\mathcal{L}_1$ and $\mathcal{L}_2$ (intuitively, a rougher and a finer version of $\mathcal{G}$), such that

$$k_1 + q_1\overline{\mathcal{R}}(\mathcal{L}_1) \leq J(P) \leq k_2 + q_2\overline{\mathcal{R}}(\mathcal{L}_2), \qquad (21)$$

where q_1, q_2, k_1 and k_2 depend on the geometric properties of $\mathcal{G}$, i.e. s, r, γ, ρ, and on $p_{\min}$ and $p_{\max}$ only, and where we recall that $\overline{\mathcal{R}}(\mathcal{L}_1)$ and $\overline{\mathcal{R}}(\mathcal{L}_2)$ denote the average effective resistances in $\mathcal{L}_1$ and $\mathcal{L}_2$ (with unitary resistances).

Due to Remark 4.1, to prove the theorem we only need to show that the following inequality holds true

$$\bar{k}_1 + \bar{q}_1\overline{\mathcal{R}}(\mathcal{L}_1) \leq \overline{\mathcal{R}}(\mathcal{G}) \leq \bar{k}_2 + \bar{q}_2\overline{\mathcal{R}}(\mathcal{L}_2). \qquad (22)$$

for another set of constants.

The power of this result lies on the fact that $J(P)$ is known for some families of highly structured matrices. In particular, in the case $\mathcal{G}_P$ is a d-dimensional grid with N vertices, we know [5], [6] that $J(P)$ grows linearly in N if $d = 1$, logarithmically if $d = 2$, and is constant if $d \geq 3$. By Remark 4.1, knowledge of $J(P)$ can be used to bound the $\bar{\mathcal{R}}(\mathcal{G}_P)$: the two will behave, up to multiplicative constants, in the same way. Thus, we can say that the average effective resistances of d-dimensional grids grow with the same behaviors as above.

The contribution of the theorem is to show that actually the same behaviors extend also to the geometric graphs. In the following section we will show how to obtain the upper bound of Eq. 22, the lower being analogous.

B. An upper bound

We tessellate the hypercube Q in order to obtain a rough approximation of $\mathcal{G}$, and then compute the bound for the effective resistance. As it is done in Fig. 1 define $\gamma_\ell := \frac{\ell}{\lfloor \frac{\ell}{\gamma} \rfloor}$, where γ is defined in Eq. 19. Consider now the (exact) partition of Q made by hypercubes of edge length $2\gamma_\ell$.

Denote by $V_\mathcal{L}$ the set of the centers of such hypercubes. We consider the lattice $\mathcal{L} = (V_\mathcal{L}, \mathcal{E}_\mathcal{L})$ where $\mathcal{E}_\mathcal{L} = \{(u_\mathcal{L}, v_\mathcal{L}) \in V_\mathcal{L} \times V_\mathcal{L} \,|\, d_{\mathrm{E}}(u_\mathcal{L}, v_\mathcal{L}) = 2\gamma_\ell\}$. Observe that $\gamma_\ell > \gamma$ but, because $\ell > \gamma$, also $\gamma_\ell \leq 2\gamma$. We denote $Q_{u_\mathcal{L}}$ the hypercube whose center is $u_\mathcal{L}$.

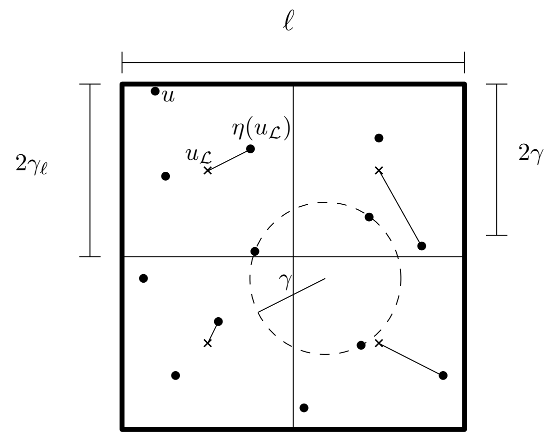

Fig. 1. The lattice $\mathcal{L}$ constructed for the upper bound. Black dots are the vertices of $\mathcal{G}$. Crosses are the centers of the squares forming the tessellation of Q, i.e., the vertices of $\mathcal{L}$. The dashed circle is the largest ball centered in Q which does not contain any vertex of $\mathcal{G}$, so that its radius defines γ.

The key idea is that this lattice is constructed in such a way that it is isomorphic to a subgraph of a suitable h-fuzz of $\mathcal{G}$. The first step is to associate vertices of $\mathcal{L}$ with a subset $\tilde{V}_\mathcal{L} \subseteq V$. By definition of γ, the construction of $\mathcal{L}$ is such that the following lemma holds true.

Lemma 5.1: For each $u_\mathcal{L} \in V_\mathcal{L}$, there exists $v \in V$ such that $v \in Q_{u_\mathcal{L}}$.

In words, every hypercube in the partition we have constructed contains at least one element of V. Let's now define a map $\eta : V_\mathcal{L} \to V$ which assigns to each $u_\mathcal{L} \in V_\mathcal{L}$ and arbitrarily chosen (but fixed) element of V contained in the corresponding hypercube $Q_{u_\mathcal{L}}$. By Lemma 5.1, this map is well-defined and injective by construction, and in general not surjective. Let $\tilde{V}_\mathcal{L} := \eta(V_\mathcal{L})$. Moreover, we define a map $\phi : V \to \tilde{V}_\mathcal{L}$ which associates to any vertex $u \in V$ the vertex (also in V) which has been selected as the representant of the hypercube in which u lies, namely $\phi(u) = \eta(u_\mathcal{L})$ if $u \in Q_{u_\mathcal{L}}$. An example of this can be seen in Fig. 1, where $\eta(u_\mathcal{L})$ is also $\phi(u)$ (not reported in the figure for clarity).

The first thing we want to obtain is an embedding of $\mathcal{L}$ in a suitable h-fuzz $\mathcal{G}^{(h)} = (V, \mathcal{E}^{(h)})$ of $\mathcal{G}$: in order to do this, we must be sure that an edge in $\mathcal{E}_\mathcal{L}$ has a correspondent in $\mathcal{E}^{(h)}$. Let's say $(u_\mathcal{L}, v_\mathcal{L}) \in \mathcal{E}_L$: a simple observation allows us to conclude that $d_{\mathrm{E}}(\eta(u_\mathcal{L}), \eta(v_\mathcal{L})) \leq 2\sqrt{d+3}\gamma_\ell$, where d is the dimension. Let's now use the parameter ρ, defined in Eq. 20.

By definition, for each pair $(u,\,v) \in V^2$,

$$\rho \leq \frac{d_{\mathrm{E}}(u,\,v)}{d_{\mathcal{G}}(u,\,v)} \Rightarrow d_{\mathcal{G}}(u,\,v) \leq \frac{1}{\rho} d_{\mathrm{E}}(u,\,v)$$

and so

$$d_{\mathcal{G}}(u,\,v) \leq \frac{2\sqrt{d+3}\gamma_\ell}{\rho} \leq \frac{4\sqrt{d+3}\gamma}{\rho}.$$

If we take $h = \lceil \frac{4\sqrt{d+3}\gamma}{\rho} \rceil$, then surely in $\mathcal{E}^{(h)}$ there exists the edge $(\eta(u_{\mathcal{L}}), \eta(v_{\mathcal{L}}))$, which is what we wanted.

Let's now bound "how far", graphically, the generic node v of V can be from $\tilde{V}_{\mathcal{L}}$. Clearly, if $v \in \tilde{V}_{\mathcal{L}}$, this graphical distance is zero. If this is not the case, let $u_{\mathcal{L}}$ be such that $v \in Q_{u_{\mathcal{L}}}$ and let $u = \eta(v)$. Because it is clear that $d_{\mathrm{E}}(v,\,u)$ is at most $2\sqrt{d}\gamma_\ell$, their graphical distance is bounded by

$$\begin{aligned} d_{\mathcal{G}}(v,\,u) &\leq \frac{1}{\rho} d_{\mathrm{E}}(v,\,u) \leq \frac{2\sqrt{d}\gamma_\ell}{\rho} \\ &\leq \frac{4\sqrt{d}\gamma}{\rho} \leq \frac{4\sqrt{d}\gamma}{\rho} = h, \end{aligned}$$

so

$$d_{\mathcal{G}^{(h)}}(v,\,u) \leq 1.$$

So, $d_{\mathcal{G}^{(h)}}(v,\,\tilde{V}_{\mathcal{L}}) \leq 1,\, \forall\, v \in V$.

Remark 5.1: This actually just means that all the nodes in the h-fuzz which are in the same $Q_{u_{\mathcal{L}}}$ are connected by an edge

What we have proved by now is that $\mathcal{L} \subseteq \mathcal{G}^{(h)}$, up to an isomorphism. Assume $u,\,v \in V$, and assume $\overline{u} = \phi(u)$ and $\overline{v} = \phi(v)$, $u_{\mathcal{L}} = \eta^{-1}|_{\tilde{V}_{\mathcal{L}}}(\overline{u})$ and $v_{\mathcal{L}} = \eta^{-1}|_{\tilde{V}_{\mathcal{L}}}(\overline{v})$. The first step in order to prove the upper bound in Eq. 22 is to recall the result from Lemma :

$$\mathcal{R}_{uv}(\mathcal{G}) \leq h\mu \mathcal{R}_{uv}(\mathcal{G}^{(h)}). \tag{23}$$

What we want to do now is to bound the average $\overline{\mathcal{R}}(\mathcal{G}^{(h)})$ making use of $\overline{\mathcal{R}}(\mathcal{L})$, and this will yield the result.

In order to do this, remember that the effective resistance is a distance, that each resistor is unitary and that $R_{uv}(\mathcal{G}^{(h)}) \leq d_{\mathcal{G}^{(h)}}(u,\,v)$. We obtain, by direct computation,

$$R_{uv}(\mathcal{G}^{(h)}) \leq 2 + R_{u_{\mathcal{L}}v_{\mathcal{L}}}(\mathcal{L}).$$

Using Eq. 23, we have thus

$$R_{uv}(\mathcal{G}) \leq 2h\mu + h\mu R_{u_{\mathcal{L}}v_{\mathcal{L}}}(\mathcal{L}). \tag{24}$$

Making use of the h-fuzz, we have thus obtained a direct bound between the effective resistance in the original graph and the effective resistance in a lattice.

Averaging now over any possible pair of nodes both in $\mathcal{G}$ and in $\mathcal{L}$, we have the upper bound in 22,

$$\overline{\mathcal{R}}(\mathcal{G}) \leq 2h\mu + h\mu\alpha\beta\overline{\mathcal{R}}(\mathcal{L}) \tag{25}$$

where

- $\alpha = \max_{u_{\mathcal{L}}} \left\{ |\phi^{-1}(\eta(u_{\mathcal{L}}))| \right\}$: this is surely bounded by the maximum number of nodes which can be put inside a square of edge length $\gamma_\ell < 2\gamma$, which does not depend on N but only on the geometric parameters of the graph;
- $\beta = \frac{|\tilde{V}_{\mathcal{L}}|^2}{N^2}$: note $|\tilde{V}_{\mathcal{L}}|$ is proportional to N by a constant related to γ. So, we can assume β to be constant in the number of nodes too.

VI. Conclusions and future work

In the paper we have shown how to relate a LQ-type cost for the consensus algorithm to the average effective resistance in a suitable network. The monotonicity properties of the effective resistances yield, under some mild assumptions, the cost only depends on the communication topology, and not on the particular way the nodes use their information. We have applied this reasoning in order to show how to bound the performance of a generic geometric graph in an hypercube with those of two lattices.

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