INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Analysis of Preemptive Periodic Real Time Systems using the (max,plus) algebra

With applications in robotics

François Baccelli , Bruno Gaujal , Daniel Simon

N°3778

October 15, 1999

THÈME 1

rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--3778--FR+ENG

Analysis of Preemptive Periodic Real Time Systems using the (max,plus) algebra

With applications in robotics

François Baccelli* , Bruno Gaujal† , Daniel Simon‡

Thème 1 — Réseaux et systèmes
Projet TRIO

Rapport de recherche n° 3778 — October 15, 1999 — 37 pages

Abstract: In this paper we present the model of a system of periodic real-time tasks with fixed priorities, preemption and synchronization, performed by a robot controler, using Marked Graphs. Then, with the help of the (max,plus) algebra, we derive simple tests to check real time constraints on those tasks such as response times and the respect of deadlines. This method takes into account precedence and synchronization constraints and is not limited to a particular scheduling policy.

Key-words: Periodic real time systems, synchronization, fixed priority preemption, marked graphs, (max,plus) algebra.

(Résumé : tsvp)

* ENS, 45 rue d'Ulm, Paris, Francois.Baccelli@ens.fr
† Loria, 615 rue du jardin Botanique, Villers lès Nancy, Bruno.Gaujal@loria.fr
‡ Inria Rhône-Alpes, 655 avenue de l'Europe, 38330 Montbonnot, Daniel.Simon@inrialpes.fr

Unité de recherche INRIA Lorraine
Technopôle de Nancy-Brabois, Campus scientifique,
615 rue de Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY (France)
Téléphone : 03 83 59 30 30 - International : +33 3 3 83 59 30 30
Télécopie : 03 83 27 83 19 - International : +33 3 83 27 83 19
Antenne de Metz, technopôle de Metz 2000, 4 rue Marconi, 55070 METZ
Téléphone : 03 87 20 35 00 - Internatională: +33 3 87 20 35 00
Télécopie : 03 87 76 39 77 - International : +33 3 87 76 39 77

Analyse de systèmes temps-réels préemptifs par l'algèbre (max,plus) et applications à la robotique

Résumé : Dans cet article, nous présentons sous forme d'un Réseaux de Petri le modèle d' un système de tâches périodiques temps-réel, avec des priorités fixes, de la préemption et de la synchronisation qui sont exécutées par le controleur d'un robot. Ensuite, avec l'aide de l'algèbre (max,plus), nous établissons des tests simples pour vérifier des contraintes temps-réels sur ces tâches, comme le calcul des temps de réponse et le respect d'échéances. Cette méthode prend en compte les contraintes de précédence et de synchronisation et n'est pas limitée à une politique d' ordonnancement particulière.

Mots-clé : Systèmes temps-réel, synchronisation, préemption à priorité fixe, graphes d'événements , algèbre (max,plus)

1 Introduction

Marked graphs can be used to model processes with highly synchronous behaviors, see [1, 5, 3] for example. Here, we focus on the study of several marked graphs which interact via certain preemption schemes.

Such systems appear in the modeling of sets of tasks performed by on-board processes in a robot. Some tasks have high priority and therefore must preempt the low priority tasks. However, all tasks, those with high priority but also those with low priority have real time constraints to meet, which are of the following type. Each task is synchronized by a clock (or by a precedent task), and is made runnable at each tick of the clock. Each task must run to completion before the next synchronization tick. The ORCCAD system, detailed in Section 2 is a good example of such a systems.

This paper gives simple answers to a series of problems with an increasing degree of difficulty, all related to the quantitative behavior of such systems.

- *(max,plus) representation.* This modeling problem can be approached under two different points of view: the *contracted time* approach is more suitable for the special case of the ORCCAD model and is presented in Section 3; the *expanded time* approach allows us to model more general problems but yields less precise answers. It is presented at the end of the paper in Section 5

- *Periodicity.* A first structural problem that we address is the periodicity issue. Under rather general assumptions, we show that the whole system reaches a periodic regime after some transient behavior.

- *Cycle time.* Another key practical problem is to compute the speed of the system once it has reached its periodic regime. When the contracted time model is valid, the answer to this question is quite simple and is given as the ratio of the cycle time without preemption by the busy period of the preemptive tasks (see section 3.2).

- *Response times.* A more precise performance measure is the response time of each task defined as the duration between the time it becomes runnable and the time of its completion. A (max,plus) representation of this quantity is given in Section 4.2 whereas a brief presentation of the (max,plus) algebra is given in appendix.

- *Real time constraints.* Finally, we give a simple test to check whether the system complies with its real time constraints, during the transient period as well as during the periodic regime (Section 4.4).

In the last part of the paper, we show that this approach can be partially generalized to an arbitrary set of Marked Graphs equipped with partial order relations between sets of transitions. In that case, we cannot always use the contracted time approach but rather expand the firing time of the transitions. The main result being the fact that the system reaches a periodic regime which does not depend on the initial conditions (see Section 5).

2 Modeling of a Real Time System: The ORCCAD Framework

This section is primarily a motivation section. We take the instance of a specific framework for real time systems, ORCCAD, and show how to model its logical and timed behavior by marked graphs. We believe that the mathematical models that we develop in this section and study later on are nevertheless generic.

ORCCAD is a software environment dedicated to the design, the verification and the implementation of advanced robotics control systems. It also allows the specification and the validation of missions to be achieved by the system[1].

Periodic and multi-rate communicating tasks are executed under the control of a classical real-time operating system, using preemption based on fixed priorities assigned to tasks.

The structure of the periodic tasks, which are called module-tasks (MTs) follows the following scheme. After initialization, an infinite loop is executed where all input ports are first read, calculations are performed from these inputs and finally results are posted on all the output ports.

2.1 Synchronization

The general idea is that a partial synchronization of such tasks can improve the performances by decreasing the computing latency. However, using too many or incorrectly specified synchronizations may lead to deadlocks or temporal inconsistencies [8]. Several kinds of synchronizations can be used on ports in order to synchronize more or less tightly the set of MTs:

- ASYN-ASYN: a communication of this type does not lead to further synchronization.

[1] A freeware simplified prototype of the software and associated documentation are available through `http://www.inrialpes.fr/iramr/pub/Orccad/orccad-eng.html`.

- SYN-SYN: each communication of this type is a *rendez-vous*; the first task to reach the *rendez-vous* (either the writer or the reader) is blocked until the second one is ready.
- ASYN-SYN: the writer runs freely and posts messages on its output ports at each period; the reader either reads the message if a new one is available or is blocked until the next message is posted.
- SYN-ASYN: symmetrical to the previous case: the reader runs freely, the writer is blocked until the next request except if a new one was posted since the last reading.

2.2 Preemptions

In addition to synchronizations, MTs may interact through another mechanism, preemption.

A control system for robotics is generally made of several calculation paths : the direct control path computing control set-points from tracking errors is often quite simple and has small computation duration. It can be activated with a fast period, thus improving the performance of the control law, e.g. reducing tracking errors or increasing robustness w.r.t. modeling errors [8]. Other tasks may be used to update some parameters of the non-linear robot model. These tasks are data-handling intensive, e.g. using trigonometric functions or matrix inversion. Their duration is usually longer than the period of the direct path. Thus they must be assigned with a low priority so that their execution is preempted by every execution of the direct path calculations. Such an example is given in section 2.6.

The whole system is run over a limited number of CPUs. All the MTs using the same CPU are ordered according to their priorities. When one MT with high priority becomes runnable and starts its calculation cycle, all MTs with lower priorities are preempted on the processor. The activity of the runnable MT with the immediately lower priority resumes where it was stopped as soon as the higher priority MT has finished its cycle of calculations.

2.3 Modeling with Petri nets

Design inconsistencies may arise in several ways. *Structural* deadlocks are due to the synchronization structure itself whatever are the numerical values of temporal attributes. In addition, preemption or badly chosen numerical values of temporal attributes like tasks period and duration may lead to *temporal inconsistencies* such as non-periodic behavior of the system, even if it is free from structural deadlocks.

Therefore, we need modeling and analysis tools to automatically check for inconsistencies in the network of synchronized MTs. Such problems have been addressed in the real-time community under very general assumptions, see for example [7]. Here, we will adopt a particular modeling tool, Petri nets which will provide a simple and efficient way to carry out inconsistency tests.

As shown in Figure 1, the sequential behavior of the simplest periodic MT (reading an input port, performing a calculation, writing to an output port) may be modeled by a Petri net with three transitions. (Of course, when a MT has multiple input and output ports, we must be careful to associate a distinct transition with each read and each write.) A fourth transition is required to activate the MT subject to the periodic awakening associated with a real time clock (RTC), also modeled by a Petri net. As we shall be further concerned with temporal analysis, we may add some temporal properties to the model to obtain a *timed Petri net*, i.e. a Petri net where durations are associated with some transitions or places. We have chosen to associate the duration [d] of the MT with the calculation step transition, and thereby assume that reading and writing are instantaneous events, i.e. events of zero duration. A crossing time $[\tau]$ is also assigned to the transition associated with the RTC (Transitions associated with non zero duration are drawn with thick lines).

Since each place has just one input transition and one output transition, the resulting Petri net is a so-called *marked graph* (or event graph).

ASYN/SYN communication between two MTs is modeled as shown in Figure 2 on the left and SYN/SYN communication is modeled as shown in Figure 2 on the right. Note that the transition associated with the periodic awakening of MT2 is no longer present, since the temporal behavior of MT2 is bound to that of MT1. Once again, the combination of the two Petri nets is a marked graph.

Note that ASYN/ASYN communication does not add synchronization constraints, i.e. MTs communicating using this protocol have disconnected models.

Also, preemption is not shown directly in the Petri net model. It will be taken into account in the equations that describe the dynamics of the system.

Thanks to structural properties of marked graphs, checking for structural deadlocks can be easily done through the analysis of the initial marking of the p-invariants of the global Petri net [8]. Studying the temporal behavior of the set of MTs is more complex: classically this can be done through a more or less exhaustive exploration of the reachability graph of the Petri net, which can be costly in time and memory. Moreover, such models usually do not take into account the effect of the real-time

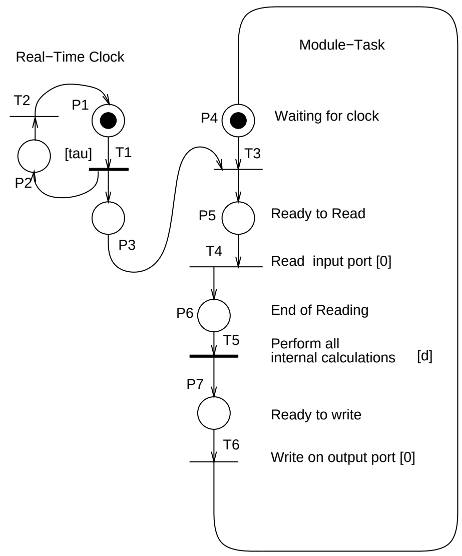

Figure 1: A Petri net model of a periodic Module-Task.

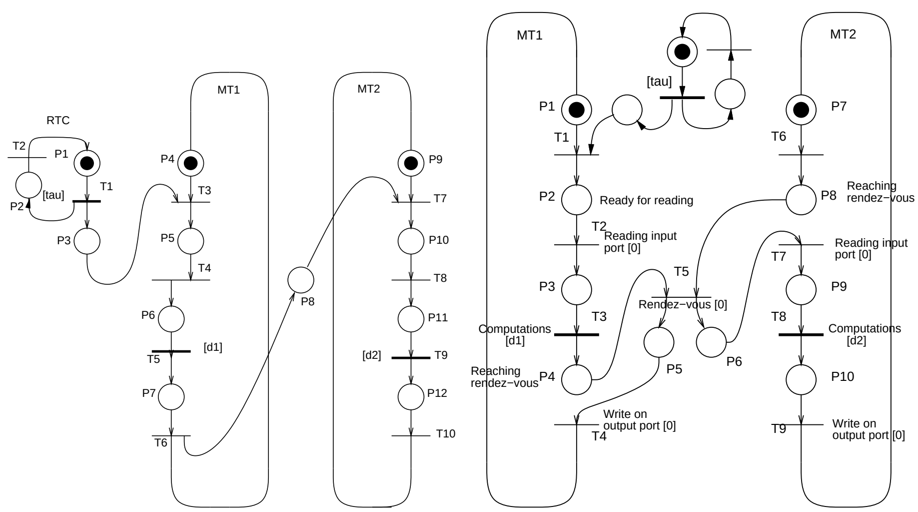

Figure 2: Petri net models for ASYN/SYN and SYN/SYN communications

scheduler which is assumed to use fixed priorities and preemption. Thus, this model must be refined accordingly, which will be the object of the present paper.

2.4 A Generic Marked Graph Model for Preemption Based Real Time Systems

We shall retain the following generic model from the above framework:

The model consists of a set of tasks T_i, $i = 1, N$. Each task T_i can be modeled by a connected marked graph $\mathcal{G}_i = (\mathcal{Q}_i, \mathcal{P}_i, M_i, \sigma_i)$,

- $\mathcal{Q}_i$ is the set of numbered transition. We denote its size Q_i;
- $\mathcal{P}_i$ is the set of numbered places. We denote its size P_i;
- $\sigma_i = (\sigma_{i_1}, \cdots, \sigma_{i_Q})$ is the set of firing times, σ_{i_q} being the firing duration of transition q. We will assume that these numbers are all multiple of the smallest time unit that can be handled by the system. Therefore, they can be seen as integer numbers.
- $M_i(r, q)$ is the initial marking in the place between r and q (if this place does not exist, $M_i(r, q)$ is not defined).
- We also denote by $q^\bullet$ and $^\bullet q$ the output places and input places respectively, of transition q.

Moreover, the strongly connected components of $\mathcal{G}_i$ are partitioned into two sets:

- The set of initial components, denoted I_i. An initial component will be called a clock. In most practical cases, this clock is always composed of a single recycled transition. However, nothing forbids to consider more elaborate clocks, and we will make no restrictive assumption concerning these initial components.
- All the other components, denoted O_i. They are often simple cycles, for single task models but may be more complicated.

As the preemption between tasks is given under the form of an order relation between the graphs $\mathcal{G}_i$. If $\mathcal{G}_j \succ \mathcal{G}_i$, whenever a transition in O_j fires, every firing transition in O_i is interrupted and resumes its firing as soon as all activities in O_j stop.

Note that the clocks of $\mathcal{G}_i$ or $\mathcal{G}_j$ are not involved in the preemption process.

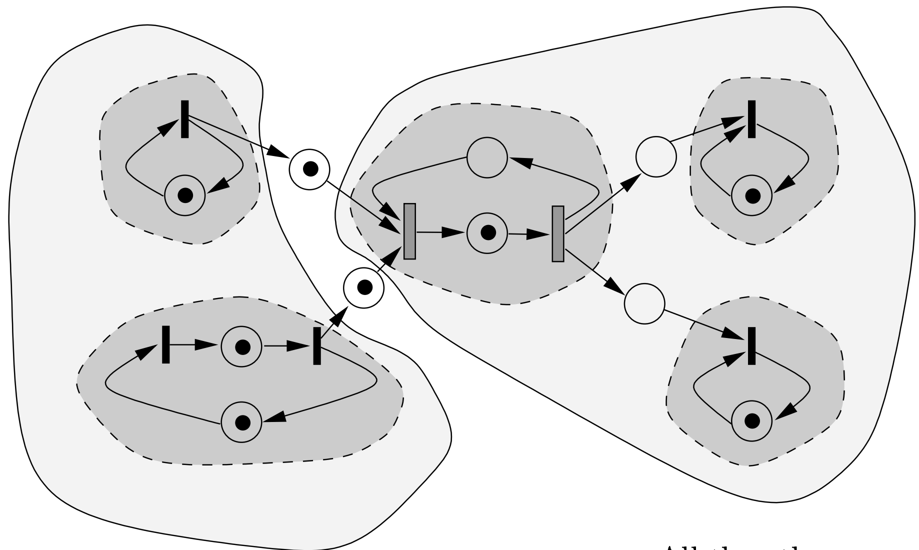

Figure 3: Example of a graph with the decomposition in the components I, O

2.5 Examples

2.5.1 Case 1: A Simple Case with no Preemption

This first example shows a group of two communicating task using a ASYN/SYN communication. However, there is no preemption in this case. This system is represented in Figure 4.

The corresponding MG is displayed in Figure 5. Each task has its own clock that sets the period of each task (20 and 15 units of time respectively).

2.6 Case 2: A case with preemption

Here, we show a realistic model of several tasks ($MT1$ to $MT7$) used in the Dynamical control in the ORCCAD system that involve preemptions. The priorities for preemptions, are such that

$$\{MT1, MT2\} \succ \{MT3, MT4, MT5, MT6\} \succ \{MT7\}.$$

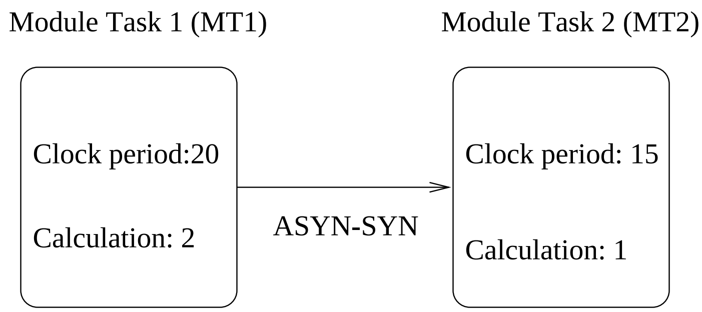

Figure 4: Two communicating tasks

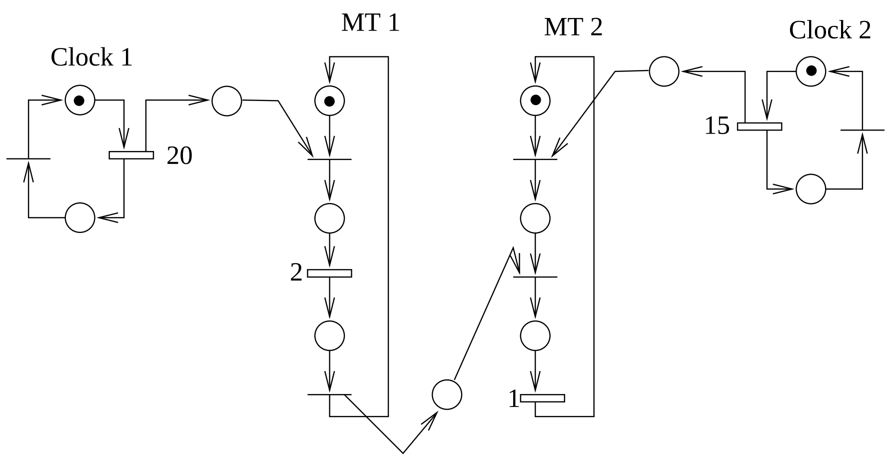

Figure 5: Corresponding Marked Graph model

Priorities have been set according to the relative urgency of the tasks. Here, MT1 and MT2 are observers checking for safety conditions, MT3–MT6 is the direct control path and the long duration MT7 computes an explicit model of the robot arm dynamics. Note that we have elected the rate-monotonic scheduling policy [4] which matches well the constraints of automatic control in that particular example (the fast direct path computation has a higher priority than the slower tasks updating parameters). Anyway, the analysis method we present in the sequel is not limited to a particular scheduling policy.

Figure 6 is the Marked Graph model of the system when there are enough processing resources available so that tasks $MT1$ and $MT2$ may occur in parallel, as well as tasks $MT3$, $MT4$, $MT5$, or $MT6$.

If there is only one processing unit (which is often the case) then tasks $MT1$ and $MT2$ must be in mutual exclusion. Similarly, tasks $\{MT3, MT4, MT5, MT6\}$ must be in mutual exclusion. As for the exclusion between the different clusters of tasks, this is taken care of by the preemption. The Marked Graph model of this system is given in Figure 7, where the activities of tasks $\{MT1, MT2\}$ are clustered into one transition, and the activities of $\{MT3, MT4, MT5, MT6\}$ are also clustered into a single transition in order to avoid useless context switches.

2.7 Problems to be addressed

The problem which this paper addresses, is to check whether all tasks will meet their time constraints, that is if each task is executed within the time slot that it is given by its clock. This general problem will be called Problem P_1 in the following. In the case with no preemption this problem is rather classical and does not require the whole machinery developed below.

A less restrictive version of this problem is to check whether violation of the time constraints may only happen a finite number of time. This is called Problem P_2 in the following.

In the marked graph models, these problems can be formulated in the following way:

- *for Problem P_1:* for all marked graph $\mathcal{G}_i$, the marking in the places that connect initial components I_i to any component O_i, is bounded by one.
- *For Problem P_2:* For all marked graph $\mathcal{G}_i$, the marking in the places that connect initial components I_i to any component O_i, may get larger than one for a finite number of occurrences.

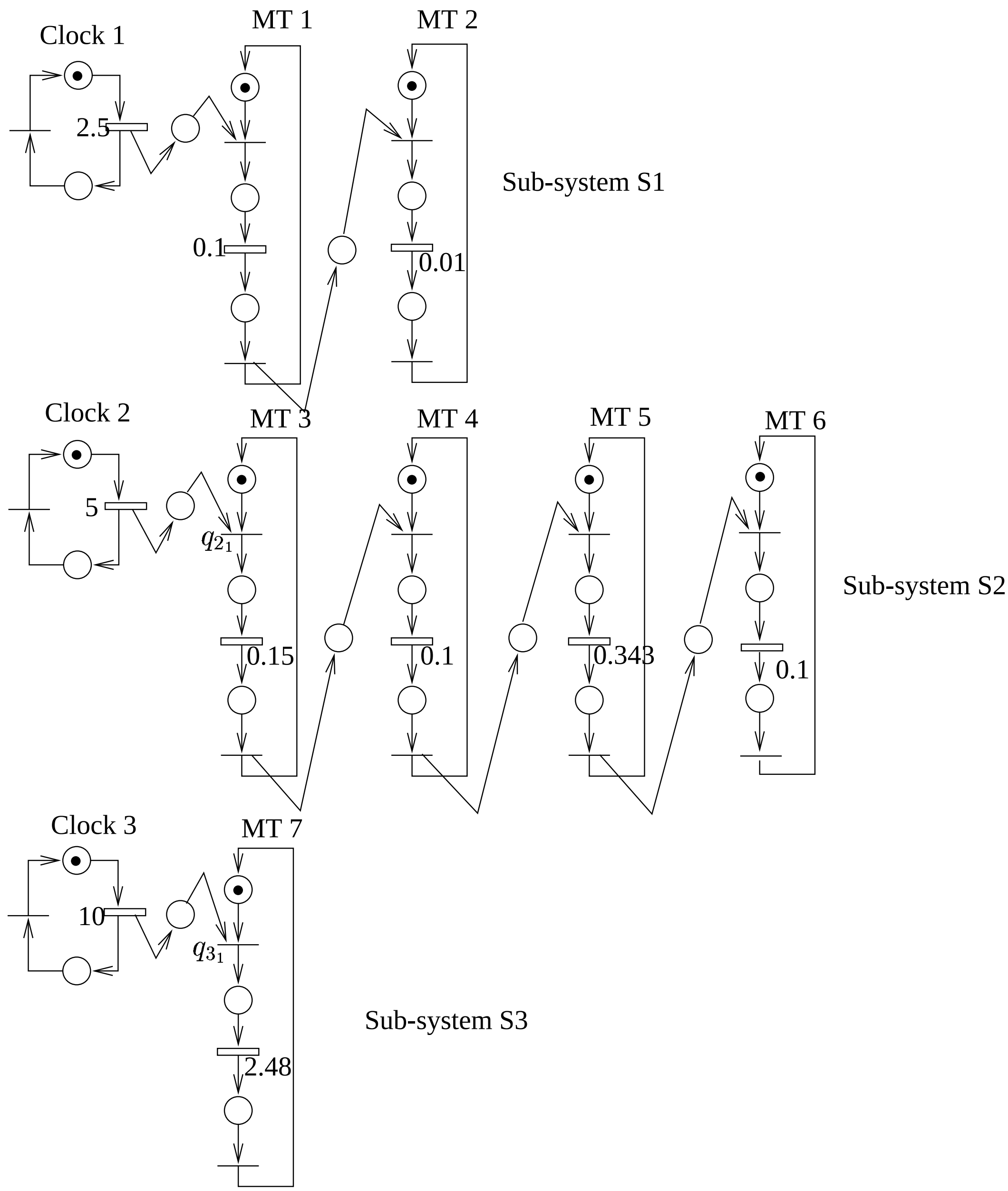

Figure 6: Marked graph model of a dynamical command, involving preemption, with several processors

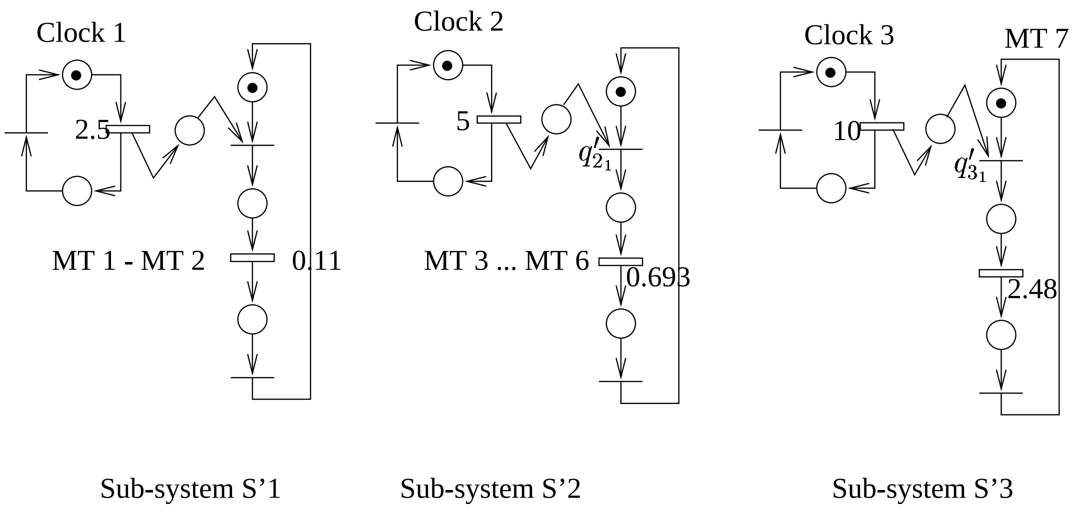

Figure 7: Marked graph model of a dynamical command, involving preemption, with one processor

The two problems P_1 and P_2 will be solved in the following section. The full solution for problems P_1 and P_2 in the cases 1 and 2 given above will then be given in Section 4.5.

3 Modeling Under Contracted Time

In this section, we introduce a representation of the system where we contract time in the non-preempted transitions. This is a way to take into account the scheduler without modeling it by a Petri net, thus allowing one to still use marked graphs.

3.1 Description of the Model

We simply consider the case where $N = 2$ which is generic, as we will see in the following.

Let $\mathcal{G}_1 \succ \mathcal{G}_2$. In the following, $\mathcal{G}_1$ will only be seen through its activity process, $S_1(t)$ that is the process which is equal to 1 if a transition in O_1 is active and 0 otherwise. It will be assumed to be periodic, with period T. As for $\mathcal{G}_2$, we will remove the index 2, since every variable considered in the following, will be for the marked graph $\mathcal{G}_2$.

We construct a copy of $\mathcal{G}(=\mathcal{G}_2)$, that we call $\mathcal{H}$ which will be considered without the preemption from $\mathcal{G}_1$.

In the following, for each Strongly Connected Component (SCC) $\mathcal{C}$, we will denote

- $\{X_q(n)\}$ the sequence of firing times of transition $q \in \mathcal{C}$, in the preempted system $\mathcal{G}$. The set of all these sequences will also be called the behavior of the system;
- $\{V_q(n)\}$ the sequence associated with the same transition in $\mathcal{H}$.

By using the theory of timed marked graphs and Lemma A.1 in the appendix, we get for all SCC $\mathcal{C}$ in isolation, a *cycle time* $\lambda_{\mathcal{C}} \in \mathbb{R}_+$, a *cyclicity* $s_{\mathcal{C}} \in \mathbb{N}_\star$ and a *transient period* $k_{\mathcal{C}} \in \mathbb{N}$, such that for all transitions $q \in \mathcal{C}$ and all $k \geqslant k_{\mathcal{C}}$,

$$V_q(k+s_{\mathcal{C}}) = V_q(k) + s_{\mathcal{C}}\lambda_{\mathcal{C}}. \tag{1}$$

Note that the activity process of this SCC is therefore periodic of period $s_{\mathcal{C}}\lambda_{\mathcal{C}}$. This period is an integer under the assumptions that we made on the firing durations.

3.2 Time Contraction

We denote by $S_1(t), t \in \mathbb{R}$ the activity process of $\mathcal{G}_1$, defined by $S_1(t) = 1$ if at time t a transition in $\mathcal{G}_1$ is active and $S_1(t) = 0$ otherwise. This function is assumed to be periodic of period T, where T is an integer.

We define $\Gamma \stackrel{\text{def}}{=} \int_0^T 1 - S_1(\tau)d\tau$ (under our assumptions, Γ is also an integer number), and $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \int_0^t 1 - S_1(\tau)d\tau.$$

Figure 8 gives a representation of S_1 and F. During each period T, $\mathcal{G}_1$ is active for Γ units of time. Also, F is *pseudo-periodic* of period T and increment Γ: *i.e.* $F(t+T) = F(t) + \Gamma$. We construct F^{-1} as the unique left continuous function such that $F(F^{-1}(t)) = t$.

Let $X'_q(k) \stackrel{\text{def}}{=} F(X_q(k))$ for all $q \in \mathcal{Q}$. By definition, we have, $X_q(n) = F^{-1}(X'_q(n))$. As shown by the following lemma, the sequences $\{X'_q(k)\}$, which give the firing times of transition q, after this time contraction, are also ultimately pseudo-periodic.

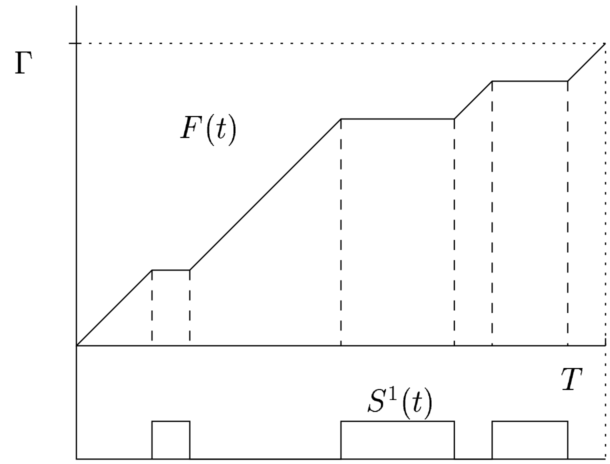

Figure 8: The contraction function F associated with an activity S_1.

Lemma 3.1. *For all SCC $\mathcal{C}$ in isolation, we have:*
i- if $\mathcal{C} \subset I$, then for all transition $q \in \mathcal{C}$ and for all $n > k_{\mathcal{C}}$, $X'_q(n + s_{\mathcal{C}}T) = X'_q(n) + s_{\mathcal{C}}\lambda_{\mathcal{C}}\Gamma$.

ii- If $\mathcal{C} \subset O$, we have all transition $q \in \mathcal{C}$ and for all $n > k_{\mathcal{C}}$, $X'_q(n + s_{\mathcal{C}}) = X'_q(n) + s_{\mathcal{C}}\lambda_{\mathcal{C}}$.

Proof. *i-* Since no transition in I is preempted, for all transitions $q \in \mathcal{C} \subset I$, $X_q(n) = V_q(n)$. Moreover, by equation (1), $V_q(n + s_{\mathcal{C}}) = V_q(n) + s_{\mathcal{C}}\lambda_{\mathcal{C}}$ for $n \geqslant k_{\mathcal{C}}$. Therefore,

$$\begin{aligned} X'_q(n + s_{\mathcal{C}}mT) &= F(X_q(n + s_{\mathcal{C}}mT)) \\ &= F(V_q(n + s_{\mathcal{C}}mT) \\ &= F(V_q(n) + s_{\mathcal{C}}mT\lambda_{\mathcal{C}}) \\ &= F(V_q(n)) + s_{\mathcal{C}}m\lambda_{\mathcal{C}}\Gamma \\ &= X'_q(n) + s_{\mathcal{C}}m\lambda_{\mathcal{C}}\Gamma. \end{aligned}$$

ii- Recall that all transitions of SCC's belonging to O are simultaneously preempted by the activity of $\mathcal{G}_1$. Therefore, in contracted time, the behavior of the every SCC in O considered in isolation is the same as in real time. □

The only difference in contracted time compared with the net in real-time is that the arrivals stemming from the SCC's of I have to be replaced by those obtained from the sequences $\{X'_q(n)\}$, $q \in I$ defined above. These arrivals are accelerated by the time change, but remain nevertheless pseudo-periodic as we just saw.

We then get from the results following Lemma A.1 in the appendix that the sequences $\{\widehat{X}'_q(n)\}$, $q \subset \mathcal{C} \in O$ which give the firing time of the global system in contracted time become ultimately pseudo-periodic. More precisely, let us define the quantities $\lambda'_{\mathcal{C}}$ as follows: if $\mathcal{C} \subset I$, then $\lambda'_{\mathcal{C}} \stackrel{\text{def}}{=} \lambda_{\mathcal{C}}\Gamma/T$. If $\mathcal{C} \subset O$, then $\lambda'_{\mathcal{C}} \stackrel{\text{def}}{=} \lambda_{\mathcal{C}}$. These quantities are the inverse of the firing rate of the transitions in contracted time. There are two cases:

1. Either the system is stable (see Appendix A.3), which happens if and only if

$$\min_{\mathcal{C} \in I} \lambda'_{\mathcal{C}} \geqslant \max_{\mathcal{C} \in O} \lambda'_{\mathcal{C}}.$$

   Let then $\mathcal{C}_0$ be a SCC such that

$$\min_{\mathcal{C} \in I} \lambda'_{\mathcal{C}} = \lambda'_{\mathcal{C}_0}.$$

   In this case, all sequences $\{\widehat{X}'_q(n)\}$ couple in finite time with a pseudo-periodic regime such that

$$\widehat{X}'_q(n + s_{\mathcal{C}_0} mT) = \widehat{X}'_q(n) + s_{\mathcal{C}_0} m \lambda_{\mathcal{C}_0}\Gamma,$$

   where m is an integer such that $mT \in \mathbb{N}$ and $m\lambda_{\mathcal{C}} \in \mathbb{N}$ for all $\mathcal{C} \in I$. In this case, the marking in all places is bounded, and both the marking and the activity processes are ultimately periodic functions of time.

2. Or the system is unstable, which happens if and only if

$$\min_{\mathcal{C} \in I} \lambda'_{\mathcal{C}} < \max_{\mathcal{C} \in O} \lambda'_{\mathcal{C}}.$$

   In this case, some places have a marking which tends to ∞. All sequences $\{\widehat{X}'_q(n)\}$ couple in finite time with a pseudo-periodic regime which we will not study here because this case corresponds to an unproper behavior of the system.

At this point, a few remarks are in order:

- The condition of stability of the whole system is stronger in the preempted case than in the non-preempted case since the stability region in this last case reads

$$\min_{\mathcal{C}\in I} \lambda_{\mathcal{C}} \leqslant \max_{\mathcal{C}\in O} \lambda_{\mathcal{C}}.$$

- The fact that the system is ultimately periodic in contracted time implies that it is also periodic in real time (as we see when applying back the function F^{-1} to the ultimately periodic sequences of interest).

- In the stable case, the activity process becomes ultimately periodic both in contracted and in real time, which allows one to proceed by induction when the number of levels N is larger that 2.

3.3 Analysis of a simple example

Consider the model of Figure 9, which is made of two SCC. The clock synchronizes the activity of the second SCC (O) and is not preempted. The second SCC is preempted by a system with an activity process with period $T = 5$ and such that $F(T) = 3$. This preemption process is similar to the activity of $\mathcal{G}_1$ in Figure 11, for example. The firing times of the clock transition is $\sigma = 4$. The firing times of the two transition in O are 2 and 1 respectively.

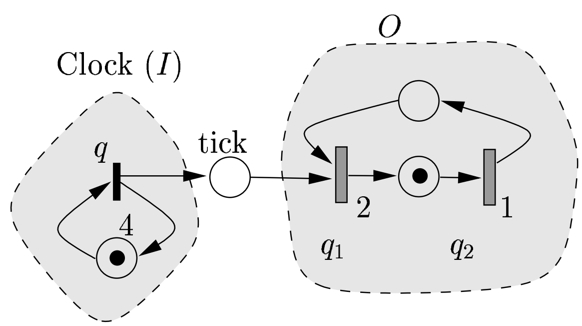

Figure 9: A clock and its synchronous circuit.

The stability condition can be tested here: first, note that with the data given in Figure 9, we have $T = 5, \Gamma = 3$. In contracted time, for the clock:

$\widehat{X}'_1(n+5) = \widehat{X}'_1(n) + 4 \times 3$

For the system O, we have: $\widehat{X}'_2(n+1) = (\widehat{X}'_2(n) \otimes 3) \oplus \widehat{X}'_1(n)$.

Therefore, although when there is no preemption the system is stable: $\lambda_1 = 4 > \lambda_2 = 3$, under preemption, we have $\lambda'_1 = 12/5 < \lambda'_2 = 3$ and the system becomes unstable. The marking of the place "tick" goes to infinity.

From the real-time computing point of view, this means that the task represented by O is restarted before completion, leading to a system failure.

4 Detailed Analysis of the Periodic Regime

In this section, we will consider further properties of the periodic regime of the system.

In application to robot tasks, one important property that the system must satisfy is a real time constraint of the following form: each task has to be executed once between two ticks of its clock.

In the marked graph model, as mentioned in Section 2.7, this corresponds to verifying whether a token produced by the clock finds an empty place (the constraint is met) or not (the previous task has not been executed before the new clock tick, and the constraint is violated).

4.1 Formulation of the problems in the (max,plus) algebra

The following arguments will be done in contracted time. We focus on a system with one initial component $\mathcal{C}_1$ in I and one component $\mathcal{C}_2$ in O. $\mathcal{C}_1$ is connected to $\mathcal{C}_2$ through a place (called "tick" in Figure 9). The output transition of place "tick" (which belongs to $\mathcal{C}_2$) is numbered q_1.

Let us denote $u(n)$ the epoch of the nth arrival of a token in place tick, in contracted time. We have $u(n) = F(t_n)$, where t_n is the date of the n-th arrival in tick in real time.

We also define $\tau(n) = u(n+1) - u(n)$.

Note that using Lemma 3.1, $\{u(n)\}$ is pseudo-periodic with period T and $u(n+T) = u(n) + \sigma\Gamma$.

As for the process of $\mathcal{C}_2$, under contracted time, it is a marked graph identical to the original marked graph without preemption (in isolation). Again, this is a consequence of Lemma 3.1). Therefore, under contracted time, the whole system is an open (max,plus) system which can be represented under the form

$$X(n) = A \otimes X(n-1) \oplus B \otimes u(n). \tag{2}$$

We will denote by n_0, c, γ the coupling, cyclicity and maximal eigenvalue of matrix A, respectively (see Appendix A.1 and A.3 where (max,plus) notations and their relations with the dynamics of marked graphs are described).

Algebraic formulations of the problems

Lemma 4.1. *Problem P_1 can be written as*

$$X_1(n) - u(n+1) < 0, \quad \forall n \geqslant 1, \tag{3}$$

where $X_1(n)$ is the n-th firing time of transition q_1 and Problem P_2 can be written as

$$X_1(n) - u(n+1) < 0, \quad \forall n \geqslant n_0, \tag{4}$$

where n_0 is the time when the periodic regime is reached.

Proof. $X_1(n)$ is the instant when transition q_1 removes the nth token in place "tick" while $u(n+1)$ is the instant when the $n+1$th token is put in place "tick" by the clock. If $X_1(n) > u(n+1)$, then there are at least two tokens in place "tick" during the interval $[u(n+1), X_1(n))$. □

In the following, we will assume that

$$X_1(1) = u(1). \tag{5}$$

This assumption is natural in our context: it means that the system is ready to start as soon as the clock emits its first signal.

Remark In the case where the clock is a simple recycled transition (as in Figure 9) with firing duration σ, we have $u(n) = F(n\sigma)$, $n \geqslant 1$. We can also give an exact representation of the whole clock under contracted time, displayed in Figure 10. This construction is simply based on the following observation. Under contracted time, the first token must arrive in place "tick" at time $F(\sigma)$; the second token at time $F(2\sigma)$, and so forth, with a period T.

This allows one to derive response times in contracted times which can be converted in real time by applying F^{-1}.

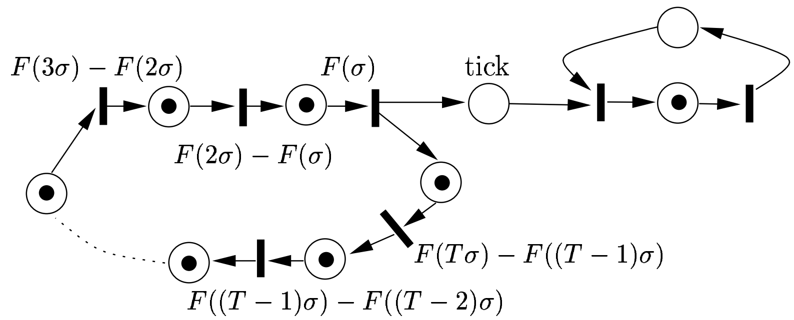

Figure 10: Representation of the clock in contracted time

4.2 Problem P_2

- The first case is when the clock cycle time $(\sigma\Gamma/T)$ is smaller than the maximal eigenvalue γ of matrix A. In this case, $X_1(n)$ is of the same order as $n\gamma$ when n goes to infinity and $u(n+1)$ is of order $n\sigma\Gamma/T$. Therefore, there exists some n_0 such that $u(n+1) > X_1(n)$ for all $n \geqslant n_0$. In this case none of the properties P_2 and P_1 are satisfied.

- Now, consider the case when $\sigma\Gamma/T \geqslant \gamma$ (Assumption **H1**). We define the vector $Z(n) = X(n) - u(n+1)$. Since $\sigma\Gamma/T \geqslant \gamma$, $X(n)$ is also ultimately pseudo-periodic with period T and cycle time $\sigma\Gamma/T$ (see Lemma 3.1).

Therefore, the variable $Z(n)$ is ultimately periodic with period T.

$$Z(kT+s) = Z((k-1)T+s), \quad \forall k \geqslant k_0, \quad \forall 0 \leqslant s \leqslant T-1. \tag{6}$$

Using the formula for $X(kT+s)$, we also have for all $k \geqslant 1$ and $0 \leqslant s \leqslant T-1$,

$$
\begin{aligned}
Z(kT+s) &= X(kT+s) - u(kT+s+1) \\
&= A \otimes X(kT+s-1) \oplus B \otimes (u(kT+s) - u(kT+s+1)) \\
&= A \otimes D(-\tau(kT+s)) \otimes Z(kT+s-1) \oplus B \otimes (-\tau(kT+s)) \\
&= A \otimes D(-\tau(kT+s)) \otimes \Big(A \otimes D(-\tau(kT+s-1)) \otimes Z(kT+s-2) \\
&\quad \oplus B \otimes (-\tau(kT+s-1)) \Big) \oplus B \otimes (-\tau(kT+s)) \\
&= A^{\otimes T} \otimes D(-\sigma\Gamma) \otimes Z((k-1)T+s) \\
&\quad \bigoplus_{i=0}^{T-1} \Big(A^{\otimes i} \otimes D(-\tau(kT+s) - \cdots - \tau(kT+s-i)) \otimes B \Big),
\end{aligned}
$$

where $D(x)$ is the diagonal matrix with x on the diagonal. All of this is true using the fact that the diagonal matrix $D(.)$ commutes with everything.

Let $M = A^T \otimes D(-\sigma\Gamma)$ and

$$
C(s) = \bigoplus_{i=0}^{T-1} A^i \otimes D(-u(kT+s+1) + u(kT+s-i)) \otimes B.
$$

Using Equation (6), we can rewrite the last relation

$$
Z(kT+s) = M \otimes Z(kT+s) \oplus C(s), \quad \forall k \geqslant k_0. \tag{7}
$$

Since the matrix M has a maximal eigenvalue which is negative by Assumption **H1**, then the matrix M^* exists (see Appendix A.2 for the definition and the condition of existence of matrix M^*) and the minimal solution of (7) is equal to

$$
Z(kT+s) = M^* \otimes C(s), \quad \forall k \geqslant k_0. \tag{8}
$$

Theorem 4.2. *Property P_2 is satisfied if and only if coordinate 1 in $M^* \otimes C(s)$ is non-positive for all $0 \leqslant s \leqslant T-1$.*

Computation Complexity The computation of $Z(kT+s)$ requires the calculation of $A^{\otimes 2}, \cdots, A^{\otimes T}$, which takes $O(T|Q|^3)$ units of time. The computation of M^* takes $O(|Q|^3)$ units of time and the computation of $C(s)$ for all s takes $O(T)$ units of time. The bottleneck will be given by the computation of $A^{\otimes 2}, \cdots, A^{\otimes T}$. Note however, that it is important to have a complexity which is linear in T, since T may be large.

4.3 Initial Phase Issues

It is often the case that the different tasks of the operating systems may have different initial phases each time the system is started anew.

Theorem 8 gives a test for Problem P_2 for a fixed given phase, however, we would like to derive a test to ensure that the time constraint is satisfied for all possible phases between the preemptive system and the preempted one. Such problems have been studied in [6] for example.

Since time is slotted, the total number of phases ϕ is finite, equal to T. A brute force formula to check problem P_2 under all possible phases is to check whether

$$\bigoplus_{\phi=0}^{T-1} \bigoplus_{s=0}^{T-1} M^* \otimes C_\phi(s),$$

has a non-positive first coordinate, with $C_\phi(s)$ defined as above with $\tau(kT+s)$ replaced by $\tau_\phi(n) \stackrel{\text{def}}{=} u_\phi(n+1) - u_\phi(n)$, where the variables $u_\phi(n) \stackrel{\text{def}}{=} F(t_n + \phi)$ are the clock ticks under phase ϕ.

However, the complexity of this formula is in $O(T^3)$, which could be prohibitive when T grows (this happens in particular when the preemption gets very complex, like in the case of the superposition of several preemptive tasks).

We now derive a better formula, by characterizing the worst possible phase between the two systems.

We consider the first coordinate of $Z(kT+s)$, and the case where B is the vector $(0, -\infty, \cdots, -\infty)$. Then

$$\begin{aligned} Z(kT+s)_1 &= \left(M^* \otimes \left(\bigoplus_{i=0}^{T-1} A^i \otimes D(-u_\phi(kT+s+1) + u_\phi(kT+s-i)) \otimes B \right) \right)_1 \\ &= \bigoplus_{i=0}^{T-1} (M^* A^i)_{1,1} \otimes (-u_\phi(kT+s+1)) \otimes u_\phi(kT+s-i). \end{aligned}$$

Now, we maximize over all s and all ϕ,

$$
\begin{aligned}
K &\stackrel{\text{def}}{=} \bigoplus_{\phi=0}^{T-1}\bigoplus_{s=0}^{T-1} Z(kT+s)_1 \\
&= \bigoplus_{i=0}^{T-1}(M^*A^i)_{1,1}\bigoplus_{\phi=0}^{T-1}\bigoplus_{s=0}^{T-1}(-u_\phi(kT+s+1))\otimes u_\phi(kT+s-i) \\
&= \bigoplus_{i=0}^{T-1}(M^*A^i)_{1,1}\bigoplus_{\phi=0}^{T-1}\bigoplus_{s=0}^{T-1}(-u_\phi(s+i+1))\otimes u_\phi(s).
\end{aligned}
$$

We choose $s^* = s^*(i)$ such that $t_{i+1+s^*} - t_{s^*} \leqslant t_{i+1+s} - t_s$ for all $0 \leqslant s \leqslant T$. We choose ϕ^* such that $F(t_{i+1+s^*} + \phi^*) - F(t_{s^*} + \phi^*) \leqslant F(t_{i+1+s^*} + \phi) - F(t_{s^*} + \phi)$ for all $0 \leqslant i \leqslant T$ and for all $0 \leqslant \phi \leqslant T$.

Now, for all i, using the fact that F is non-decreasing, we have

$$
\begin{aligned}
-u_{\phi^*}(s^*+i+1) + u_{\phi^*}(s^*) &= -F(t_{i+1+s^*}+\phi^*) + F(t_{s^*}+\phi^*) \\
&\geqslant -F(t_{i+1+s}+\phi) + F(t_s+\phi) \\
&= -u_\phi(s+i+1) + u_\phi(s),
\end{aligned}
$$

for all s and ϕ. Therefore, the worst case is reached with $s = s^*$ and $\phi = \phi^* = \phi^*(i)$. Checking Problem P_2 for all possible phases to done by checking whether

$$
K = \bigoplus_{i=0}^{T-1}(M^*A^i)_{1,1} \otimes (-u_{\phi^*}(s^*+i+1)) \otimes u_{\phi^*}(kT+s^*)
$$

is non-positive.

Complexity Note that the construction of the worst phases for the clock as well as for the preemption process has to be done for all possible values of i. For a fixed i, this computation can be done independently of the rest. The computation of s^* takes $O(T)$ units of time, as well as the computation of ϕ^*.

Therefore, checking problem P_2 for all possible phases can be done in $O(T^2)$ units of time instead of $O(T^3)$ with a the brute force computation.

4.4 Transient Issues: Problem P_1

In this section, we show how to check property P_1. For this we have to study the transient period.

We recall that A is ultimately pseudo-periodic with coupling time n_0 and period c. The process $\tau(n)$ is ultimately periodic with coupling time m_0 and period T. The vector $\mathbf{e}$ is the vector of size $|\mathcal{I}|$ with each component equal to 0.

We will derive an upper bound on the length of the transient period of the global system. Then, the verification of property (P_1) can be done simply by computing the behavior of the system up to that bound and checking (P_1) at each step.

To derive this upper bound, we use the variable $W(n) = X(n) - u(n)$ which satisfies an equation similar to Equation (7)

$$W(n+1) \quad = \quad A \otimes D(-\tau(n)) \otimes W(n) \oplus B, \quad \forall n \geqslant 1. \tag{9}$$

Note that $W(n) \geqslant 0$ for all n. We set $H(m,n) = A^{\otimes(n-m)} \otimes D(-\tau(n) - \cdots - \tau(m+1))$ if $m < n$ and $F(n,n) = Id$.

Lemma 4.3. *For all $m \geqslant 0$, $H(m,n)$ has coupling time $t_0 = \max(n_0, m_0)$ and is pseudo-periodic in n with period $p = lcm(T,c)$, where c is the cyclicity of A.*

Proof. Indeed, for all $n \geqslant t_0$,

$$\begin{aligned} H(m,n+p) &= A^{\otimes(n+p-m)} \otimes D(-\tau(n+p) - \cdots - \tau(n+1)) \\ &\quad \otimes D(-\tau(n) - \cdots - \tau(m+1)) \\ &= (\gamma p/c - \sigma\Gamma p/T) \otimes A^{\otimes(n-m)} \otimes D(-\tau(n) - \cdots - \tau(m+1)) \\ &= (\gamma p/c - \sigma\Gamma p/T) \otimes H(m,n). \end{aligned}$$

Note that this is true for all values of m. □

As for $W(n)$, we get for all $k > 1$,

$$\begin{aligned} W(t_0+kp) &= H(0,t_0+kp) \otimes W(1) \oplus \bigoplus_{i=1}^{t_0+kp} H(i,t_0+kp) \otimes B \\ &= k(\gamma p/c - \sigma\Gamma p/T) \otimes \mathbf{e} \otimes H(0,t_0) \otimes W(1) \oplus \bigoplus_{i=1}^{t_0+kp} H(i,t_0+kp) \otimes B. \end{aligned}$$

Choosing

$$k \geqslant \beta_1 \stackrel{\text{def}}{=} \frac{H(0,t_0) \otimes W(1)}{-\gamma p/c + \sigma\Gamma p/T} + 1, \tag{10}$$

which is positive, we have

$$(k-j)(\gamma p/c - \sigma\Gamma p/T) \otimes \mathbf{e} \otimes H(0,t_0) \otimes W(1) \leqslant 0, \quad j=0,1. \tag{11}$$

And choosing

$$k \geqslant \beta_2 \stackrel{\text{def}}{=} \frac{\mathbf{e} \otimes \bigoplus_{i=1}^{p+t_0} H(i, t_0+p) \otimes B}{-\gamma p/c + \sigma\Gamma p/T} + 1, \tag{12}$$

we also have

$$H(i, t_0 + (k-j)p) \otimes B \leqslant 0, \quad \forall 1 \leqslant i \leqslant p+t_0, \quad j=0,1. \tag{13}$$

Therefore, for k larger than $\beta_1 \vee \beta_2$, we get

$$\begin{aligned}
W(t_0+kp) &= \bigoplus_{i=1}^{t_0+kp} H(i, t_0+kp) \otimes B && (14)\\
&= \bigoplus_{i=p+t_0+1}^{t_0+kp} H(i, t_0+kp) \otimes B && (15)\\
&= \bigoplus_{i=t_0+1}^{t_0+(k-1)p} H(i, t_0+(k-1)p) \otimes B && (16)\\
&= W(t_0+(k-1)p), && (17)
\end{aligned}$$

where Equation (14) comes from Inequality (11), Equation (15) comes from Inequality (13), both with $j=0$, and (16) from the fact that $H(i, t_0+kp) = H(i-p, t_0+(k-1)p)$, for all $i \geqslant p+t_o$. Equation (17), comes from the fact that using (10) and (12), Inequalities (11) and (13) are also valid for $j=1$.

Finally, $W(t_0+kp) = W(t_0+(k-1)p)$ means that W has reached its periodic regime before step $t_0+(k-1)p$). Once W has reached its period, then this means that all the system also reached its periodic regime.

Theorem 4.4. *The periodic regime is reached after a transient period of length at most:*

$$t_0 + \frac{\mathbf{e} \otimes H(0,t_0) \otimes (W(1)) \oplus \bigoplus_{\alpha=1}^{p+t_0} H(\alpha, t_0+p) \otimes B}{-\gamma/c + \sigma\Gamma/T}.$$

where $W(1)$ has to be computed, using the initial conditions of the system.

Note that t_0 can be replaced by a larger bound (but easier to compute):

$$t_0 + \frac{\mathbf{e} \otimes (I \oplus A)^{2t_0+p} \otimes (B \oplus W(1))}{-\gamma/c + \sigma\Gamma/T}.$$

4.5 Application to the Examples of § 2.5

Case 1 can be solved by a simple application of Lemma A.1, because it does not contain preemption. This is a simple marked graph. In that case neither P_1 nor P_2 are satisfied. Indeed, task MT1 is executed every 20 units of time. The ASYN/SYN communication between MT1 and MT2 imposes that task MT2 will eventually be executed every 20 units of time. However, it has a clock period requirement of 15 units of time, which will not be met from its second execution on. This is a case in which Assumption **H1** does not hold.

Case 2, with parallel tasks (as given in Figure 6) contains preemption.

- The first thing to do is check whether the cycle times of all tasks are all well ordered so that the system is stable (Assumption **H1**).

  The sub-system (S1) made of clock 1, MT1 and MT2 is stable : $(2.5 > 0.1$ and $2.5 > 0.01)$.

  Now, we consider the second connected component, made of Clock 2, $MT3$, $MT4$, $MT5$ and $MT6$ (S2). This sub-system is preempted by the first component, which has an activity of period 2.5, with $T = 2.5$ and $\Gamma_2 = 2.39$.

  Therefore, for the stability of the second component, we have to check that $5 \times \frac{2.39}{2.5} > 0.15, \quad 5 \times \frac{2.39}{2.5} > 0.1, \quad 5 \times \frac{2.39}{2.5} > 0.343$, which is true.

  The last component (S3) is preempted by both sub-systems S1 and S2. The whole preemption process has period $T = 5$, and a total non busy time of $\Gamma_3 = 5 - ((0.1 + 0.01) \times 2 + 0.15 + 0.1 + 0.343 + 0.1) = 4.087$.

  The stability property becomes: $10 \times \frac{4.087}{5} > 2.48$. We can conclude that the whole system is stable.

- The second test is to check whether property P_2 is satisfied. We will apply Theorem 4.2 for all sub-systems.

  Sub-system (S1) satisfies P_2 because its period is one and because it is stable.

  Sub-system (S2) is preempted by sub-system 1. However, it also has period 1 (in terms of number of firings). The input under contracted time is: $u(n) =$

$u(n-1)+\Gamma_2$, with $\Gamma_2 = 4.88$. Its structure can be reduced to a scalar version of Equation (2): $x_{2_1}(n) = a_2 \otimes x_{2_1}(n-1) \oplus u(n)$, with $a_2 = 0.1$.

In that case, we obtain

$$z_2(k+1) = x_{2_1}(k) - u(k+1) = (a_2 - \Gamma_2) \otimes z_2(k) \oplus -\Gamma_2,$$

the maximal solution of which is $z_2(k+1) = (a_2 - \Gamma_2)^* \otimes -\Gamma_2$, once the periodic regime of z_2 is reached (here, when $k > 1$).

The numerical solution is $z_2(k+1) = -\Gamma_2 = -4.22$, which is negative.

As for sub-system (S3), we get similarly, a periodicity equal to 1, and a solution $z_3(k) = (a_3 - \Gamma_3)^* \otimes -\Gamma_3$ with $\Gamma_3 = -8.174$ and $a_3 = 2.48$. The solution is $z_3(k) = -8.174$. Therefore, Property P_2 is satisfied in the three sub-systems.

- Finally, one has to consider the transient regime of all marked graphs, to verify problem $P1$. Here, all systems have period 1 (in number of firings) as well as a transient regime of length 1. The periodic regime is reached immediately and property P_1 is verified without using Theorem 4.4.

Case 2, in the sequential case (as given in Figure 7) also contains preemption. The analysis is similar to the previous case.

- cycle times:

  The sub-system (S'1) made of clock 1, MT1-MT2 is stable : $(2.5 > 0.11)$.

  Now, we consider the second connected component (S'2), made of Clock 2, MT3- ... -MT6. This sub-system is preempted by the first component, which has an activity of period 2.5, with $T = 2.5$ and $\Gamma_2 = 2.39$. Stability of (S'2) holds since $5 \times \frac{2.39}{2.5} > 0.693$.

  The last component (S'3) is preempted by both sub-systems S'1 and S'2. The whole preemption process has period $T = 5$, and a total non busy time of $\Gamma_3 = 5 - ((0.1 + 0.01) \times 2 + 0.15 + 0.1 + 0.343 + 0.1) = 4.087$.

  The stability property holds: $10 \times \frac{4.087}{5} > 2.48$. The whole system is stable.

- Property P_2:

  Sub-system (S'1) satisfies P_2 as in the parallel case.

  Sub-system (S'2) is preempted by sub-system 1. However, it also has period one(in terms of number of firings). The input under contracted time is: $u(n) =$

$u(n-1)+\Gamma_2$, with $\Gamma_2 = 4.88$. Its structure can be reduced to a scalar version of Equation (2): $z_2(k+1) = x_{2_1}(k) - u(k+1) = (a_2 - \Gamma_2) \otimes z_2(k) \oplus -\Gamma_2$, with $a_2 = 0.11$. The solution is $z_2(k+1) = (a_2 - \Gamma_2)^* \otimes -\Gamma_2 = -\Gamma_2 = -4.21 < 0$, once the periodic regime of z_2 is reached.

As for sub-system (S'3), we get similarly, a periodicity equal to 1, and a solution $z_3(k) = (a_3 - \Gamma_3)^* \otimes -\Gamma_3$, with $a_3 = 2.48$. and $\Gamma_3 = -8.174$. The solution is $z_3(k) = -8.174 < 0$.

- Transient regime:

  All systems have period 1 (in terms of number of firings) as well as a transient regime of length 1. The periodic regime is reached immediately and property P_1 is also satisfied immediately.

5 A More General Model

Some of the qualitative results which where established in the previous sections (periodicity) can actually be shown for more general models. However quantitative results do not extend easily to the most general models.

5.1 A Refined Preemption scheme

The model presented in Section 2 can be generalized by making the preemption more general by choosing a subset of preempting transitions in $\mathcal{G}_i$ and a subset of preemptable transitions in $\mathcal{G}_i$ which are completely general (and not limited to the set O_i and O_j). More precisely, to each marked graph $\mathcal{G}_i$, we associate $\mathcal{P}_i$ which is the set of preemptive transitions and $\mathcal{R}_i$ which is the set of preemptable transitions. Now, if $\mathcal{G}_j \succ \mathcal{G}_i$, then no transition in $\mathcal{R}_i$ may fire at the same time as a transition in $\mathcal{P}_j$. Note that the previous model corresponds to the case where $\mathcal{P}_j = O_j$ and $\mathcal{R}_i = O_i$. One can even imagine that the sets $\mathcal{P}_i$ and $\mathcal{R}_i$ depend on the couple (i,j), in which case we would denote $\mathcal{P}_{(j,i)}$ the set of preemptive transitions in $\mathcal{G}_j$ over $\mathcal{G}_i$ and $\mathcal{R}_{(i,j)}$ the set of transitions in $\mathcal{G}_i$ preempted by $\mathcal{G}_j$. This case is not treated in the following of the paper but only requires extra technical details to be handled.

An example of such a system with two graphs $\mathcal{G}_1 \succ \mathcal{G}_2$ is given in Figure 11. $\mathcal{P}_1 = \{q_{1_2}\}$ and $\mathcal{R}_2 = \{q_{2_2}\}$. As for the firing times, we choose $\sigma_{1_1} = 3, \sigma_{1_2} = 2$ and $\sigma_{2_1} = 4, \sigma_{2_2} = 2, \sigma_{2_3} = 1$.

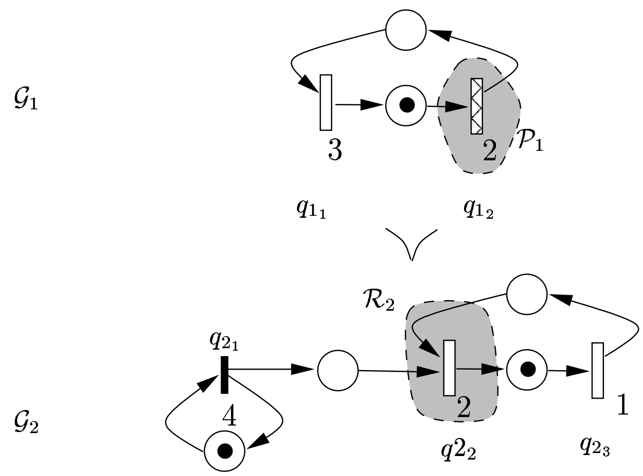

Figure 11: A system with $\mathcal{G}_1 \succ \mathcal{G}_2$.

5.2 Behavior of the System

If $q \in \mathcal{G}_i$, we denote by $X_{i_q}(n)$ the epoch when transition q starts its n-th firing.

For every graph $\mathcal{G}_i$, we define the isolated version of $\mathcal{G}_i$, denoted $\mathcal{H}_i$, in which no transitions are ever preempted. More formally, $\mathcal{H}_i$ is a version of $\mathcal{G}_i$ with $\mathcal{R}_i = \emptyset$. The behavior of $\mathcal{H}_i$ is denoted $U_i(n)$. In the event graph $\mathcal{H}_i$ with constant firing times (σ_{i_q}), the variables $U(n) = (U_1(n), \cdots, U_Q(n))$ satisfy an evolution equation of the form (see [1]),

$$U_{i_q}(n) = \max_{r \in {}^\bullet q} \left(U_{i_r}(n - M_{(r,q)}) + \sigma_{i_r} \right). \tag{18}$$

We also know from [1] that Equation (18) has the following property: there exists n_{i_0}, k_{i_q} and λ_{i_q} such that for $n \geqslant n_{i_0}$,

$$U_{i_q}(n) = U_{i_q}(n - k_{i_q}) + \lambda_{i_q} k_{i_q}.$$

In Figure 12, the functions $S_1(t)$ as well as $X_{2_1}(t)$ and $X_{2,2}(t)$ are displayed. Note that since transition q_{2_1} is not preempted, and is not dependent on the other transitions in $\mathcal{Q}_2$, its behavior is not affected and $X_{2_1}(t) = U_{2_1}(t)$ for all t.

However, the behavior of q_{2_2} is affected by the preemption. Note that the functions X and U verify, $X_{2_2}(t) \leqslant U_{2_2}(t)$.

Also note that if $\mathcal{G}_i \not\succ \mathcal{G}_j$, then the behavior of $\mathcal{G}_i$ does not depend on the behavior of $\mathcal{G}_j$. As a consequence, if $\mathcal{G}_i$ is not preempted by any other graph, its behavior can

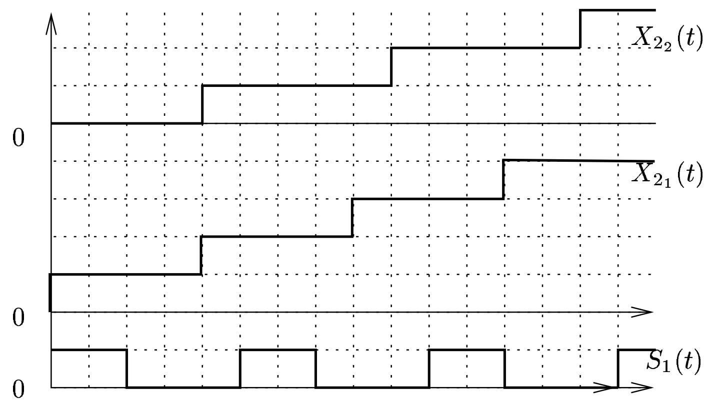

Figure 12: the activity of $\mathcal{G}_1$ and the behavior of $\mathcal{G}_2$.

be determined in isolation, that is $\forall q \in \mathcal{Q}_i, n \in \mathbb{N}, X_{i_q}(n) = U_{i_q}(n)$. In this case, we have the following property: there exists n_{i_0}, k_{i_q} and λ_{i_q} such that for $n \geqslant n_{i_0}$, $X_{i_q}(n) = X_{i_q}(n - k_{i_q}) + \lambda_{i_q} k_{i_q}$.

Definition 5.1. *The sequence of expanded firing times δ_{i_q} of a transition q in $\mathcal{G}_i$ are defined as follows:*
If $q \in \mathcal{R}_i$, then

$$\delta_{i_q}(x) = \inf \left\{ u \ : \ \int_x^{x+u} \mathbf{1}_{\{S_j(t)=0, j \succ i\}} dt \geqslant \sigma_{i_q} \right\}. \tag{19}$$

If $q \notin \mathcal{R}_i$, then for all x,

$$\delta_{i_q}(x) = \sigma_{i_q}. \tag{20}$$

This definition implies that $(\mathcal{G}_i, \sigma_i)$ has the same dynamical behavior of $(\mathcal{H}_i, \delta_i)$ in the following sense:

$$X_{i_q}(n) = \max_{r \in \bullet q} X_{i_r}(n - M_i(r,q)) + \delta_{i_r}\big(X_{i_r}(n - M_i(r,q))\big). \tag{21}$$

This equation looks like a classical (max,plus) equation describing the dynamic of a marked graph, the only difference here being that the firing times depend on the current state.

5.3 Qualitative Analysis for $N = 2$

In the following, we will keep the assumption that time is slotted. The slot duration will be the time unit and all durations will be multiples of this unit.

We first study the case where $N = 2$, $\mathcal{G}_1 \succ \mathcal{G}_2$ and $\mathcal{P}_1$ and $\mathcal{R}_2$ are not empty.

Therefore, for n large enough and for all transitions $q \in \mathcal{Q}_1$, $X_{1_q}(n) = X_{1_q}(n - k_{1_q}) + \lambda_{1_q} k_{1_q}$.

Let $T \stackrel{\text{def}}{=} \text{lcm}_{q \in \mathcal{Q}_1}(\lambda_{1_q} k_{1_q})$. The activity $S_1(t)$ is a function which becomes eventually periodic with period T after a transient period of $n_1 \stackrel{\text{def}}{=} \max_q X_{1_q}(t_0)$ units of time.

Lemma 5.2. *The behavior of transition q in $\mathcal{G}_2$ is either finite (q fires a finite number of times) or pseudo-periodic with period $p_q k_{2_q}$. In the latter case, there exists integers a and p_q and μ such that $\forall q \in \mathcal{G}_i$, $\forall n \geqslant a$,*

$$X_{2_q}(n + p_q k_{2_q}) = X_{2_q}(n) + \mu T.$$

Proof. If transition q fires an infinite number of times, then the preempting activity is not always equal to 1 in its period. This implies that the mapping $x \rightarrow \delta_{2_q}(x)$ is bounded.

Now, let $k_2 \stackrel{\text{def}}{=} \text{lcm}_{q \in \mathcal{Q}_2} k_{2_q}$. We consider the marking $M(nk_2T)$ in all places at time nk_2T, $n > \max(n_{2_0}, n_{1_0})$, as well as the residual firing time vector at time nk_2T, denoted $R(nk_2T)$. Since $\delta_{2_q}(x)$ is bounded and integer valued for integer x, and since all the involved quantities are integer numbers, then there exist integers $a > b > \max(n_{0_2}, n_{0_1})$ such that $M(ak_2T) = M(bk_2T)$ and $R(ak_2T) = R(bk_2T)$.

The whole process is such that the preemption as well as the initial condition are the same at times ak_2T and bk_2T. Therefore, the system evolves periodically. This implies that for each transition q $X_{2_q}(n + p_q k_{2_q}) = X_{2_q}(n) + \mu T$, for all $n > ak_2T$, where $p_q k_{2_q}$ is the number of firings of q between times ak_2T and bk_2T, and $\mu = bk_2 - ak_2$. □

The following technical lemma will be useful in the proof Theorem 5.4, which is the main result of this section.

Lemma 5.3. *Let Φ be a component-wise non-decreasing function from $\mathbb{R}_+^Q \rightarrow \mathbb{R}_+^Q$, Then, if $\gamma \stackrel{\text{def}}{=} \lim_{n\to\infty} \Phi^{(n)}(x_0)/n$ exists and $\gamma > 0$, then, for all $x \geqslant x_0$,*

$$\lim_{n\to\infty} \Phi^{(n)}(x)/n = \gamma.$$

Proof. Since $\gamma > 0$, then the sequence $\Phi^{(n)}(x_0)$ is increasing to infinity. Therefore, for all $x \geqslant x_0$, there exists integers n and m such that $\Phi^{(n)}(x_0) \leqslant x \leqslant \Phi^{(m)}(x_0)$. Therefore, $\Phi^{(n+k)}(x_0) \leqslant \Phi^{(k)}(x) \leqslant \Phi^{(m+k)}(x_0)$ for all k. We finish the proof by letting k go to infinity. □

Theorem 5.4. *There exists a constant γ_q such that*

$$\lim_{n\to\infty} X_{2_q}(n)/n = \gamma_q.$$

Moreover, this constant does not depend on the initial conditions, $X_1(0)$ and $X_2(0)$.

Proof. The sequence $X_{2_q}(n)$ is pseudo-periodic and non-decreasing. Therefore, the above limit γ exists and $\gamma = \frac{\mu T}{p_q k_{2_q}}$. Furthermore, the function δ_{2_r} given by the evolution equation (21) does not depend on n since the firing times of transitions in $\mathcal{H}_2$ are constant. We have the following evolution equation in this framework:

$$X_{2_q}(n) = \max_{r\in\bullet q} X_{2_r}(n - M_2(r,q)) + \delta_{2_r}(X_{2_r}(n - M_2(r,q))). \tag{22}$$

For all x, the function $x \to x + \delta_{2_r}(x)$ is non-decreasing in x, since the integral is taken over a non-negative function. Therefore, the function

$$\Phi : \mathbb{N}^Q \quad \to \quad \mathbb{N}^Q \tag{23}$$

$$X \quad \to \quad \left(\max_{r\in\bullet q} X_r + \delta_{2_r}(X_r)\right), \tag{24}$$

is component-wise non-decreasing. Also note that

$$X_{2_q}(n) = \Phi_q(X_{2_1}(n - M_2(1,q)), \cdots, X_{2_Q}(n - M_2(Q,q))).$$

By using lemma 5.3, we know that the limit of $X_{2_q}(n)/n$ does not depend on the initial value of $(X_1(0), X_2(0))$. □

Within the graph $\mathcal{G}_2$, we can distinguish the behavior of the different strongly connected components.

Corollary 5.5. *If two transitions, q and r belong to the same strongly connected component in $\mathcal{G}_2$, then they have the same cycle time: $\gamma_q = \gamma_r$ and the same period: $p_q = p_r$.*

Proof. Suppose that $\gamma_q \neq \gamma_r$. Then we have $|X_{2_q}(n) - X_{2_r}(n)| \to \infty$. This means that the marking in a path between transitions q and r is unbounded, which is impossible in a strongly connected component. Moreover, if r and q are in the same component, then $k_{2_q} = k_{2_r}$. Therefore, the cycle period is equal for both of them, in which case we get: $p_q = p_r$. □

5.4 Generalization to arbitrary N

We consider the case where N is arbitrary as well as $\succ$. We further assume that each marked graph $\mathcal{G}_i$ is strongly connected.

The general case can be reduced the case $N = 2$ when using the following arguments. When one studies a marked graph $\mathcal{G}_i$,

- all graphs $\mathcal{G}_j, i \succ j$ are not taken into account.
- All graphs $\mathcal{G}_j, i \succ j$ are aggregated into a single activity process, $\Theta(t) = \max_{j \succ i} S_j(t)$ with period $T = \text{ lcm } T_j$.
- All the proofs are done by induction w.r.t. the order $\succ$.

Therefore, we have the following results by directly using these arguments and the results obtained for $N = 2$.

Theorem 5.6. *For all i, $X_i(n)$ is pseudo-periodic. The cycle time, $\lim_{n\to\infty} X_i(n)/n$ (denoted γ_i) does not depend on the initial conditions, $(X_1(0), \cdots, X_N(0))$.*

Corollary 5.7. *If $\mathcal{G}_i$ is strongly connected, The expanded firing times of $\mathcal{G}_i$ are pseudo-periodic.*

Proof. The proof holds by induction with the order $\succ$. Assume that X_j, δ_j is cyclic, then S_j is periodic with period T_j. By Lemma 5.2, X_i is pseudo-periodic with period T_i. Therefore, the expanded firing times δ_i are periodic using the expression for expanded firing times given in (19). □

Critical Graph In general, we can note that the critical graph of $\mathcal{H}_i$ is different from the critical graph of $\mathcal{G}_i$ in general. However, this is not the case when $\mathcal{R}_j = \mathcal{Q}_j$. Determining the critical graph of $\mathcal{G}_i$, (and hence the length of the period) as well as the influence of the initial conditions are still open problems.

Homogeneity and Monotonicity Using the representation given in Equation (21), it is not difficult to see that the global state of the whole system, $X = (X_1, \cdots, X_N)$ is monotone and homogeneous.

We define $M \stackrel{\text{def}}{=} \max\{M_i(r,q), r,q \in \mathcal{Q}_i, \quad i \in \{1,N\}\}$, the maximal marking in any place of the whole system. We construct the global evolution function

$$\begin{array}{llcl} \Psi: & (\mathbb{N}^{Q_1} \times \cdots \times \mathbb{N}^{Q_N})^M & \to & \mathbb{N}^{Q_1} \times \cdots \times \mathbb{N}^{Q_N} \\ & ((X_1(n-1), \cdots, X_N(n-1)) & & \\ & \vdots & \to & (X_1(n), \cdots, X_N(n)) \\ & (X_1(n-M), \cdots, X_N(n-M)) & & \end{array}$$

Since the function $\delta_r(x)$ is increasing in x for all r, then the function Ψ is component-wise increasing.

Also, by construction, Ψ is homogeneous, that is $\Psi(X+h) = \Psi(X) + h$.

Therefore, all the results concerning topical functions can be used in this case, see [2].

6 Summary

In this paper we have developed a new technique to analyze the quantitative temporal behavior of a set of periodic tasks. We assume that the tasks are scheduled using preemption and fixed priorities, and that their deadline equals their period. The model also takes into account synchronizations between tasks enforcing precedence constraints. Under these assumptions, the set of tasks can be modeled by Timed Marked Graphs which have a linear model in the (max,+) algebra.

Using this model we derived tests to check some temporal properties of the system such as periodicity, cycle time, response time and respect of deadlines, both for the transient regime and for the steady state regime. The method is quite general and is not limited to a particular scheduling policy like rate monotonic thus leaving freedom to choose the priority assignment according to, e.g., automatic control performance constraints.

The technique presented here was applied to the particular case of the orccad environment, however, it is rather general and can be used in a more general framework of systems with preemption and fixed priority. As for problems with shared resources, potentially involving priority inversions this is not currently tackled and needs some further investigations.

A Appendix: (max,plus) algebra and marked graphs

In this section we will list several properties of the so-called (max,plus) algebra. All of these results can be found in [1], where they are presented in full details.

A.1 The (max,plus) semi-ring

$\mathbb{R}_{max}$ is the semi-ring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $\oplus$ stands for the max operation and $\otimes$ stands for the $+$ operator.

These operations are extended to vectorial operation in the canonical way, If A and B are matrices with appropriate dimensions on $\mathbb{R}_{max}$, then $C = A \oplus B$ is a matrix with $C_{ij} = A_{ij} \oplus B_{ij}$ and $D = A \otimes B$ is a matrix with $D_{ij} = \bigoplus_k A_{ik} \otimes B_{kj}$. For simplicity, $A \otimes A$ will be denoted $A^{\otimes 2}$ or A^2. When a is a scalar and A a matrix, in $\mathbb{R}_{max}$, then $a \otimes A$ is a matrix with each component equal to $a \otimes A_{ij}$.

A.2 Elements of spectral theory

Let A be an irreducible matrix over $\mathbb{R}_{max}$, then there exists a *coupling time* $n_0 \in \mathbb{N}$, a *cyclicity* $c \in \mathbb{N}$, a unique *eigenvalue* $\lambda \in \mathbb{R}$ and at least an *eigenvector* v such that

$$A \otimes v = \lambda \otimes v \tag{25}$$

and

$$\forall n \geqslant n_0, A^{\otimes(n+c)} = \lambda^{\otimes c} \otimes A^{\otimes n}. \tag{26}$$

If A is an irreducible matrix with a non-positive eigenvalue, then, the equation $X = A \otimes X + b$, where X is the unknown vector and b is a fixed vector, admits a unique finite solution $X = A^* \otimes B$, where A^* is the finite matrix:

$$A^* \stackrel{\text{def}}{=} \bigotimes_{i=0}^{\infty} A^{\otimes i}. \tag{27}$$

A.3 Marked Graphs

To any marked graph $\mathcal{G}$ with an initial marking bounded by one, one can associate matrices, $A(k)$, $k \in \{0, 1\}$, of size $Q \times Q$, where the entry (i, j) in matrix $A(k)$ is σ_j, the delay or lag time of transition j, if there exists a place between transitions q_j and q_i with k initial tokens, and $-\infty$ otherwise.

Let $A(0)^* = \bigoplus_{i=0}^{\infty} A(0)^i$, and $A = A(0)^* \otimes A(1)$. Let $X_q(k)$ be the epoch when the k-th firing starts in transition q. Then if there are no input transitions, the Q-dimensional vectors satisfy the recurrence relation

$$X(n) = A \otimes X(n-1), \quad n \geqslant 1$$

If there is an input transition with arrival process u, where $u(n)$ gives the epoch of the nth release of a token by the input, then

$$X(n) = A \otimes X(n-1) \oplus B \otimes u(n),$$

where $B_i = 0$ if there is a place between the input and transition q and $-\infty$ otherwise.

By using the spectral theory with timed marked graphs, we get the following result.

Lemma A.1. *For all SCC $\mathcal{C}$ in isolation, there exists a cycle time $\lambda_{\mathcal{C}} \in \mathbb{R}_+$, a cyclicity $s_{\mathcal{C}} \in \mathbb{N}_\star$ and a transient period $k_{\mathcal{C}} \in \mathbb{N}$, such that for all transitions $q \in \mathcal{C}$ and all $k \geqslant k_{\mathcal{C}}$,*

$$V_q(k + s_{\mathcal{C}}) = V_q(k) + s_{\mathcal{C}}\lambda_{\mathcal{C}}. \tag{28}$$

As for the whole system $\mathcal{G}$ (all SCC considered together), in the case with no input, we have the following result : we denote $\mathcal{C} \to \mathcal{C}'$ if the SCC $\mathcal{C}$ precedes the SCC $\mathcal{C}'$ for the topological ordering.

If

$$\max\{\lambda_{\mathcal{C}} | \mathcal{C} \to \mathcal{C}'\} > \lambda_{\mathcal{C}'}, \tag{29}$$

then the SCC $\mathcal{C}'$ has the same cycle time as the preceding SCC achieving the maximum in Equation (29).

If

$$\max\{\lambda_{\mathcal{C}} | \mathcal{C} \to \mathcal{C}'\} < \lambda_{\mathcal{C}'}, \tag{30}$$

Then the SCC $\mathcal{C}'$ (and hence the whole system) is said unstable (the marking in some places will grow to infinity).

A similar result holds for a marked graph with a pseudo periodic input u. In particular, if the inverse of the input rate is larger than or equal to the maximal cycle time of all SCCs in isolation, then the system is stable. Otherwise, it is unstable.

References

[1] F. Baccelli, G. Gohen, G.J. Olsder, and J.-P. Quadrat. *Synchronization and Linearity*. Wiley, 1992.

[2] Jean Cochet-terrasson, Stéphane Gaubert, and Jeremy Gunawardena. Dynamics of min-max functions. Technical Report HPL-BRIMS-97-13, HP Laboratories Technical Report, 1997.

[3] B. Gaujal. *Parallélisme et simulation des systèmes à événements discrets.* PhD thesis, Univ. Nice, 1994. In french.

[4] C.L. Liu and J.W. Layland. Scheduling algorithms for multiprogramming in hard real-time environment. *Journal of the ACM*, 20(1):40–61, February 73.

[5] Jean Mairesse. *Stabilité des systèmes à événements discrets stochastiques, approche algébrique.* PhD thesis, École polytechnique, Palaiseau, France, 1995.

[6] J.M. Migge and A. Jean-Marie. Timing analysis of real-time scheduling policies: a trajectory based model. Technical Report 3561, INRIA, 1998.

[7] Jörn Migge. *Real-Time Scheduling: a trajectory based model.* PhD thesis, University of Nice, France, 1999.

[8] D. Simon, E. Castillo, and P. Freedman. Design and analysis of synchronization for real-time closed-loop control in robotics. *IEEE Trans. on Control Systems Technology*, july 1998.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399