Passivity-based switching control of flexible-joint complementarity mechanical systems

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Abstract

In this study one considers the tracking control problem of a class of nonsmooth Lagrangian systems with flexible joints and subject to frictionless unilateral constraints. The task under consideration consists of a succession of free-motion and constrained-motion phases. A particular attention is paid to impacting and detachment phases. A passivity-based switching controller that allows to extend the stability analysis described in our previous works to the case of systems with lumped flexibilities, is proposed. Numerical tests show the effectiveness of the controller.

Key words: Flexible joints, Passivity-based control, Nonsmooth systems, Lagrangian systems, Complementarity problem.

1 Introduction

The control of systems undergoing impacts has received attention in the literature Albu-Schaffer *et al* (2004); Lee *et al* (2003); Pagilla (2001, 2004); van Vliet *et al* (2000); Xu *et al* (2000). In parallel with such works focusing solely on the collision phase, more general studies concerning the stability and tracking control of nonsmooth unilaterally constrained mechanical systems have been published Bentsman and Miller (2007b); Bourgeot and Brogliato (2005); Brogliato *et al* (1997); Galeani *et al* (2008); Leine and van de Wouw (2008a,b); Menini and Tornambè (2001a,b); Miller and Bentsman (2006); Tornambé (1999); Yu and Pagilla (2006). Until now these works have been limited to perfectly rigid systems. The consideration of flexibilities is important. On one hand impacts may damage systems with too small flexibility, whereas flexibility can reduce the damage by impact absorption Wolf and Hirzinger (2008). On the other hand, impact phenomena may excite vibrational modes, which is not desirable in practice and may destabilize the system (see Section 7) when the flexibilities are too large. Introducing flexibility however is challenging for the control design. In this work it is shown that the tracking control framework developed in Bourgeot and Brogliato (2005); Brogliato *et al* (1997); Morărescu and Brogliato (2008) can be adapted to the flexible-joint case, using the passivity-based motion control solutions proposed in Brogliato *et al* (1995). More precisely, this paper focuses on the problem of tracking control of complementarity Lagrangian systems Moreau (1988), encompassing flexible-joint manipulator subject to frictionless unilateral constraints, whose dynamics is supposed to be expressed as:

$$\begin{cases} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + K(q-\theta) = D^\top \lambda \\ J\ddot{\theta} + K(\theta - q) - K\mathcal{Z}(q,\theta) = U \\ q^1 \geq 0,\ (q^1)^T\lambda = 0,\ \lambda \geq 0 \\ \text{Collision rule} \end{cases} \tag{1}$$

where $q \in \mathbb{R}^n$ is the vector of rigid links angles, $\theta \in \mathbb{R}^n$ is the vector of motor shaft angles, $M(q) = M^T(q) \in \mathbb{R}^{n\times n}$ is a positive definite inertia matrix, $C(q,\dot{q})$ is the matrix containing Coriolis and centripetal forces, $G(q)$ contains conservative forces, $\lambda \in \mathbb{R}^m$ is the vector of contact forces (or Lagrangian multipliers) associated to the constraints, $J \in \mathbb{R}^{n\times n}$ is the diagonal and constant matrix of actuator inertia, $K = K^\top > 0$, $K \in \mathbb{R}^{n\times n}$ represents the stiffness matrix, $U \in \mathbb{R}^n$ is the vector of generalized torque inputs, and $q^1 = Dq \in \mathbb{R}^m$ with $D = [I_m\ 0_{m\times(n-m)}]$. A constraint i is said to be *active* if $q_i^1 = 0$, and *inactive* if $q_i^1 > 0$. The dynamics in (1) is a simplified dynamics obtained from more general Lagrangian systems using a generalized coordinate transformation as in McClamroch and Wang (1988), that is supposed to hold globally in the configuration space. Notice that a nonlinear stiffness $K\mathcal{Z}(q,\theta)$ may appear due to the transformation. Details on the transformation may be found in Morărescu *et al* (2008).

General notations and definitions. $||\cdot||$ is the Euclidean

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norm, $b_p \in \mathbb{R}^p$ and $b_{n-p} \in \mathbb{R}^{n-p}$ are the vectors formed with the first p and the last $n-p$ components of $b \in \mathbb{R}^n$, respectively. $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ represent the smallest and the largest eigenvalues, respectively. The time-derivative of a function $f(\cdot)$ is denoted by $\dot{f}(\cdot)$. For any function $f(\cdot)$ the limit to the right at the instant t will be denoted by $f(t^+)$ and the limit to the left will be denoted by $f(t^-)$ when they exist. A simple jump of the function $f(\cdot)$ at the moment $t = t_\ell$ is denoted $\sigma_f(t_\ell) = f(t_\ell^+) - f(t_\ell^-)$. For a real valued function $f : \mathbb{R}^+ \mapsto \mathbb{R}$ one denotes by $S(f)$ the set of all real valued function $g : \mathbb{R}^+ \mapsto \mathbb{R}$ such that there exists a positive real constant $0 < c < \infty$ satisfying $g(t) \leq cf(t),\ \forall t \geq 0$. One writes $g \in S(1) \equiv L^\infty$ if $f(t) = 1,\ \forall t \geq 0$. 0_n is the n-vector with entries 0, and $0_{n\times m}$ is the $n \times m$-zero matrix. I_m is the $m \times m$ identity matrix A vector is considered positive if all its component are positive. A Linear Complementarity Problem (LCP) with unknown λ is a system: $\lambda \geq 0$, $A\lambda + b \geq 0$, $\lambda^\top(A\lambda + b) = 0$, which is compactly rewritten as $0 \leq \lambda \perp A\lambda + b \geq 0$. Such an LCP has a unique solution for all b if and only if A is a P-matrix Facchinei and Pang (2003). Positive definite matrices (not necessarily symmetric) are P-matrices.

The admissible domain associated to the system (1) is the closed set $\mathbf{\Phi} \triangleq \{(q,\theta) \in \mathbb{R}^{2n} \mid q^1 \geq 0\} = \left(\bigcap_{1\leq i\leq m} \Phi_i\right) \times \mathbb{R}^n$ where $\Phi_i = \{q \in \mathbb{R}^n \mid q_i^1 \geq 0\}$. In the sequel $\left(\bigcap_{1\leq i\leq m} \Phi_i\right)$ will be denoted by $\Phi \subset \mathbb{R}^n$. Notice that $m > 1$ allows both simple impacts (when one constraint is involved) and multiple impacts (when several constraints are involved). Let us introduce the following notion of p_ϵ-impact.

Definition 1 *Let $\epsilon \geq 0$ be a fixed real number. We say that a p_ϵ-impact occurs at the instant t if*

$$||(q_i^1)_{i\in I}(t)|| \leq \epsilon, \quad \prod_{i\in I} q_i^1(t) = 0$$

where $I \subset \{1,\dots,m\}$, $card(I) = p$.

If $\epsilon = 0$ all p surfaces $\Sigma_i = \partial\Phi_i = \{q \in \mathbb{R}^n \mid q_i^1 = 0\}$, $i \in I$ are struck simultaneously. When $\epsilon > 0$ the system collides $\partial\Phi$ in a neighborhood of the intersection $\bigcap_{i\in I} \Sigma_i$.

A collision (or restitution) rule is a relation between the post-impact and the pre-impact velocities. Among the various models of collision rules, Moreau's rule is an extension of Newton's law which is energetically consistent Mabrouk (1998) and is numerically tractable Acary and Brogliato (2008). In the special coordinates of (1) this reads as $\dot{q}_i^1(t^+) = -e\dot{q}_i^1(t^-)$ when $q_{1,i}(t) = 0$ and $\dot{q}_i^1(t^-) \leq 0$, where $i \in \{1,...,m\}$ and $e \in [0,1]$. Under mild conditions on the data, the solutions are such that positions $q(\cdot)$ and $\theta(\cdot)$ are absolutely continuous functions of time, whereas the generalized velocity is right continuous of local bounded variation. Well-posedness results may be found in Dzonou and Marques (2007); Mabrouk (1998) and references therein. The continuity of $\dot{\theta}(\cdot)$ holds Brogliato (1999) and will be used in the stability analysis developed in section 6.

The structure of the paper is as follows: in Section 2 one presents some basic concepts and necessary prerequisites. Section 3 is devoted to the controller design. In this section one also defines the desired (or "exogenous") trajectories entering the dynamics. The desired contact-force that must occur on the phases where the motion is persistently constrained, is explicitly defined in Section 4. Section 5 focuses on the strategy for take-off at the end of the constraint phases. The main results related to the closed-loop stability analysis are presented in Section 6. A numerical example obtained with the SICONOS platform and concluding remarks end the paper.

2 Basic concepts

2.1 Typical task

Since the system's dynamics does not change when the number of active constraints decreases one gets the following typical task representation:

$$\mathbb{R}^+ = \bigcup_{k\geq 0}\left(\Omega_{2k}^{B_k} \cup I_k^{B_k} \cup \left(\bigcup_{i=1}^{m_k} \Omega_{2k+1}^{B_{k,i}}\right)\right) \tag{2}$$
$$B_k \subset B_{k,1};\ B_{k+1} \subset B_{k,m_k} \subset B_{k,m_k-1} \subset \dots B_{k,1}$$

where the superscript B_k represents the set of active constraints ($B_k = \{i \in \{1,\dots,m\} \mid F_i(X) = 0\}$) during the corresponding motion phase, and $I_k^{B_k}$ denotes the transient between two Ω_k phases when the number of active constraints increases. We note that $B_k = \emptyset$ corresponds to free-motion. When the number of active constraints decreases no transition phases are needed, thus, for the sake of simplicity and without any loss of generality we replace $\bigcup_{i=1}^{m_k} \Omega_{2k+1}^{B_{k,i}}$ by $\Omega_{2k+1}^{B'_k}$ and the typical task representation simplifies as:

$$\mathbb{R}^+ = \bigcup_{k\geq 0}\left(\Omega_{2k}^{B_k} \cup I_k^{B_k} \cup \Omega_{2k+1}^{B'_k}\right) \tag{3}$$
$$B_k \subset B'_k, \quad B_{k+1} \subset B'_k$$

Since the tracking control problem involves no difficulty during the Ω_k-phases, *the central issue is the study of the passages between them (the design of transition phases I_k and detachment conditions), and the stability of the trajectories evolving along (3)* (i.e. an infinity of cycles). Throughout the paper, the sequence $\Omega_{2k}^{B_k} \cup I_k^{B_k} \cup \Omega_{2k+1}^{B'_k}$ will be referred to as the cycle k of the system's evolution.

2.2 System properties

For kinematic chains with prismatic or revolute joints the following properties hold.

Property 1 *The matrix* $\left[\frac{d}{dt}M(q)\right] - 2C(q,\dot{q})$ *is skew-symmetric and* $\dot{M}(q) \triangleq \frac{d}{dt}M(q) = C(q,\dot{q}) + C^\top(q,\dot{q})$. *Furthermore the matrix* $C(q,\dot{q})$ *is a smooth function of* q *and* $\dot{q}$ *with the well-known properties* $||C(q,\dot{q})|| \in S(||\dot{q}||)$ *and* $C(q,y)z = C(q,z)y, \quad \forall\, q,y,z \in \mathbb{R}^n$.

Property 2 *The conservative forces vector* $G(q)$ *is such that* $\left|\left|\frac{\partial G(q)}{\partial q}\right|\right| \in S(1)$ *which implies by the mean value theorem* $||G(q_1) - G(q_2)|| \in S(||q_1 - q_2||), \quad \forall q_1, q_2 \in \mathbb{R}^n$.

Property 3 *The matrix* $C(q,\dot{q})$ *is such that* $\left|\left|\frac{\partial C(q,\dot{q})}{\partial q}\right|\right| \in S(||\dot{q}||)$ *and* $\left|\left|\frac{\partial C(q,\dot{q})}{\partial \dot{q}}\right|\right| \in S(1)$.

2.3 Stability analysis criteria

The system (1) is a complex nonsmooth and nonlinear dynamical system. A stability framework for this type of systems has been proposed in Brogliato *et al* (1997) and extended in Bourgeot and Brogliato (2005); Morărescu and Brogliato (2008). This is an extension of the Lyapunov second method adapted to closed-loop mechanical systems with unilateral constraints. Since we use this criterion in the following tracking control strategy it is worth to clarify the framework and to introduce some definitions. Let us define Ω as the complement in $\mathbb{R}_+$ of $I = \bigcup_{k\geq 0} I_k^{B_k}$ and assume that the Lebesgue measure of Ω, denoted $\lambda[\Omega]$, equals infinity. Let $x(\cdot)$ be the state of the closed-loop system in (1) with some feedback controller $U(q,\dot{q},\theta,\dot{\theta},t)$.

Definition 2 (Weakly Stable System) *The closed loop system is called weakly stable (Bourgeot and Brogliato (2005)) if for each* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that* $||x(0)|| \leq \delta(\epsilon) \Rightarrow ||x(t)|| \leq \epsilon$ *for all* $t \geq 0$, $t \in \Omega$. *The system is asymptotically weakly stable if it is weakly stable and* $\lim_{t\in\Omega,\, t\to\infty} x(t) = 0$. *Finally, practical weak stability holds if there exists* $0 < R < +\infty$ *and* $t^* < +\infty$ *such that* $||x(t)|| < R$ *for all* $t > t^*$, $t \in \Omega$.

Consider $I_k^{B_k} \triangleq [\tau_0^k, t_f^k]$ and $V(\cdot)$ such that there exists strictly increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying the conditions: $\alpha(0) = 0$, $\beta(0) = 0$ and $\alpha(||x||) \leq V(x,t) \leq \beta(||x||)$. In the sequel, we consider that for each cycle the sequence of impact instants $\{t_\ell^k\}_{\ell\geq 0}$ has an accumulation point t_∞^k.

Proposition 1 (Weak Stability) *Assume that the task admits the representation (3) and that*

a) $\lambda[I_k^{B_k}] < +\infty, \quad \forall k \in \mathbb{N}$,
b) *outside the impact accumulation phases* $[t_0^k, t_\infty^k]$ *one has* $\dot{V}(x(t),t) \leq -\gamma V(x(t),t)$ *for some constant* $\gamma > 0$,
c) *the system is initialized on* Ω_0 *such that* $V(\tau_0^0) \leq 1$,
d) $V(t_\infty^k) \leq \rho^* V(\tau_0^k) + \xi$ *where* $\rho^*, \xi \in \mathbb{R}_+$.

Then $V(\tau_0^k) \leq \delta(\gamma,\xi)$, $\forall k \geq 1$ *where* $\delta(\gamma,\xi)$ *is a function that can be made arbitrarily small by increasing either the value of* γ *or the length of the time interval* $[t_\infty, t_f]$. *Thus, the system is practically weakly stable with* $R = \alpha^{-1}(\delta(\gamma,\xi))$.

PROOF. From assumption **(b)** one has

$$V(t_f^k) \leq V(t_\infty^k)e^{-\gamma(t_f^k - t_\infty^k)}$$

and using condition **(d)** and **(c)** we arrive at

$$V(t_f^k) \leq e^{-\gamma(t_f^k - t_\infty^k)}(\rho^* + \xi) \triangleq \delta(\gamma,\xi)$$

Assumption **(b)** also guarantees that $V(\tau_0^{k+1}) \leq V(t_f^k)$ and thus $V(\tau_0^{k+1}) \leq \delta(\gamma,\xi)$, $\forall k \geq 1$. The term $\delta(\gamma,\xi)$ can be made as small as desired increasing either γ or the length of the interval $[t_\infty^k, t_f^k]$. The proof is completed by the relation $\alpha(||x||) \leq V(x,t)$, $\forall x,\, t$.

It is worth to point out the local character of the stability criterion in Proposition 1. This is firstly due to condition **(c)** and secondly by the synchronization constraints of the control law and the motion phase of the system (see (3) and (4)-(5) below). The weak stability relies on *almost non-increasing functions*, as introduced in Brogliato *et al* (1997) (see also Casagrande*et al* (2008)). Condition **(d)** means that the impacts may be considered as a kind of disturbance that can be suitably upper bounded. This is certainly the most crucial point in Proposition 1.

3 Tracking control framework

Throughout the paper, the following trajectories will play a role in the closed-loop dynamics:

- $q^{\mathrm{nc}}(\cdot)$ denotes the desired trajectory that the system should track if there were no constraints. We suppose that $q^{1,nc}(t)) < 0$ for some t, otherwise the problem reduces to the tracking control of a system with no constraints.
- $q_d^*(\cdot)$ denotes the signal entering the control input and playing the role of the desired trajectory during some parts of the motion.
- $q_d(\cdot)$ represents the signal entering the Lyapunov function $V(\cdot)$. This signal is set on the boundary $\partial\Phi$ after the first impact of each cycle.

These signals may coincide on some time intervals as we shall see later. Let us remind that $\tilde{\psi} = \begin{pmatrix}\tilde{q}\\ \tilde{\theta}\end{pmatrix} = \psi - \psi_d$ and introduce the notations: $s_1 = \dot{\tilde{q}} + \gamma_2\tilde{q}$, $s_2 = \dot{\tilde{\theta}} + \gamma_2\tilde{\theta}$, $s =$

$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, $\dot{q}_r = \dot{q}_d - \gamma_2 \tilde{q}$, $\overline{q} = q - q_d^*$ and $\overline{s}_1 = \dot{\overline{q}} + \gamma_2 \overline{q}$, where $\gamma_2 > 0$ is a scalar gain and $\psi_d = \begin{pmatrix} q_d \\ \theta_d \end{pmatrix}$.

3.1 *Controller design*

The tracking problem is solved using a generalization of the controller proposed in (Brogliato *et al*, 1995, Equ. (28)) and the closed-loop stability analysis of the system is based on Proposition 1. the controller is defined by

$$\begin{cases} U = J\ddot{\theta}_r + K(\theta_d - q_d) - \gamma_1 s_2 - K\mathcal{Z}(\psi) \\ \theta_d = q_d + K^{-1}U_r \end{cases} \tag{4}$$

where U_r is given by:

$$U_r = \begin{cases} U_c^{\emptyset} \triangleq U_{nc} = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - \gamma_1 s_1 \\ \qquad\qquad\qquad\qquad \text{for } t \in \Omega_{2k}^{\emptyset} \\ U_c^{B_k} = U_{nc} - P_d + K_f(P_q - P_d) \text{ for } t \in \Omega_k^{B_k} \\ U_c^{B_k} \quad \text{for } t \in I_k^{B_k} \text{ before the first impact} \\ U_t^{B_k} = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - \gamma_1 \overline{s}_1 \\ \qquad\qquad \text{for } t \in I_k^{B_k} \text{ after the first impact} \end{cases} \tag{5}$$

where $\gamma_1 > 0$ is a scalar gain, $K_f > 0$, $P_q = D^T\lambda$ and $P_d = D^T\lambda_d$ is the desired contact force during the persistently constrained motion. It is clear that during $\Omega_k^{B_k}$ not all the constraints are active and, therefore, some components of λ and λ_d are zero. Notice that on impacting phases no force feedback is applied. Also U is a function of q, θ, $\dot{q}$, $\dot{\theta}$ only (no acceleration feedback).

The closed-loop error dynamics on $\Omega_{2k}^{\emptyset}$ is given by:

$$\begin{cases} M(q)\dot{s}_1 + C(q,\dot{q})s_1 + \gamma_1 s_1 + K(\tilde{q} - \tilde{\theta}) = 0 \\ J\dot{s}_2 + \gamma_1 s_2 + K(\tilde{\theta} - \tilde{q}) = 0 \end{cases}$$

The rationale behind the change of structure of U_r after the first impact, is that it facilitates the calculation of some upper-bounds which are necessary to recast the closed-loop stability analysis into Proposition 1 (see section 6 and the Appendix).

In order to prove the stability of the closed-loop system (1) (4) (5) we will use the following positive definite function:

$$V(t,s,\tilde{\psi}) = \frac{1}{2}s_1^T M(q)s_1 + \frac{1}{2}s_2^T J s_2 + \gamma_1\gamma_2\tilde{q}^T\tilde{q} + \gamma_1\gamma_2\tilde{\theta}^T\tilde{\theta} + \frac{1}{2}(\tilde{q} - \tilde{\theta})^T K(\tilde{q} - \tilde{\theta}) \tag{6}$$

One of the difficulties of the flexible-joint case, compared with the rigid case, is that the jumps in the function $V(\cdot)$ in (6) are less easy to characterize. Indeed the terms $\theta_d(\cdot)$ and $\dot{\theta}_d(\cdot)$ are designed from a backstepping procedure and cannot be given arbitrary values, contrarily to other desired trajectories. The calculations of various upper bounds (see the Appendix) are consequently intricate.

3.2 *Design of the exogenous trajectory*

We consider that the unconstrained desired trajectory $q^{\text{nc}}(\cdot)$ can be split into two parts, one of them belonging to the admissible domain (inner part) and the other one outside the admissible domain (outer part). Throughout the paper we consider $I_k^{B_k} = [\tau_0^k, t_f^k]$ where τ_0^k is chosen by the designer as the start of the transition phase $I_k^{B_k}$ and t_f^k is the end of this phase. During the transition phases the system must be stabilized on the intersection of some surfaces Σ_i. This will be done by mimicking the behavior of a ball falling on the ground under gravity. Therefore all the components except the ones that are normal to the constraints belonging to B_k will be frozen. Moreover for robustness reasons one avoids a tangential approach and imposes some impacts defining a exogenous signal q_d^* that violates the constraints. In the sequel we deal with the tracking control strategy when the trajectory $q_d(\cdot)$ is constructed such that:

(i) when no activated constraint the orbit of $q_d(\cdot)$ coincides with the orbit of $q^{\text{nc}}(\cdot)$ and $\dot{q}_d(\tau_0^k) = 0$,

(ii) when $p \le m$ constraints are active, its orbit coincides with the projection of the outer part of $q^{\text{nc}}(\cdot)$ on the surface of codimension p defined by the activated constraints.

In order to simplify the presentation we introduce the following notations (where all superscripts $(\cdot)^k$ will refer to the cycle k of the system motion):

- t_0^k is the first impact during the cycle k,
- t_∞^k is the accumulation point of the sequence $\{t_\ell^k\}_{\ell \ge 0}$ of the impact instants during the cycle k ($t_f^k \ge t_\infty^k$),
- τ_1^k will be explicitly defined later and represents the instant when the exogenous signal q_d^* reaches a given value chosen by the designer in order to impose a closed-loop dynamics with impacts during the transition phases,
- t_d^k is the desired detachment instant.

It is noteworthy that t_0^k, t_∞^k, t_d^k are state-dependent whereas τ_1^k and τ_0^k are exogenous and imposed by the designer.

3.3 *Design of $q_d^*(\cdot)$ and $q_d(\cdot)$ during the phases $I_k^{B_k}$*

During the impacting transition phases the system must be stabilized on $\partial\Phi$. Obviously, this does not mean that all the constraints have to be activated (i.e. $q_i^1(t) = 0$, $\forall i = 1,\ldots,m$). Let us consider that only the first p constraints (eventually reordering the coordinates) define the border of

Φ where the system must be stabilized. The signal $q_d^*(\cdot)$ will be then defined as follows:

- choosing $\nu > 0$ and denoting $t' = \frac{t-\tau_0^k}{\tau_1^k-\tau_0^k}$, the components $\left(q_d^i\right)^*$, $i = 1, \ldots, p$ of $(q_d^*)_p$ are defined as:

$$\left(q_d^i\right)^*(t) = \begin{cases} a_3(t')^3 + a_2(t')^2 + a_0, & t \in [\tau_0^k, \min\{\tau_1^k; t_0^k\}] \\ -\nu V^{1/3}(\tau_0^k), & t \in (\min\{\tau_1^k; t_0^k\}, t_f^k] \end{cases} \tag{7}$$

  where $V(\cdot)$ is defined in (6) and τ_1^k is chosen by the designer such that the limit conditions $\left(q_d^i\right)^*(\tau_1^k) = -\nu V^{1/3}(\tau_0^k)$, $\left(\dot{q}_d^i\right)^*(\tau_1^k) = 0$ hold, which allows the computation of the previous coefficients as:

$$\begin{aligned} a_3 &= 2[\left(q^i\right)^{\text{nc}}(\tau_0^k) + \nu V^{1/2}(\tau_0^k)] \\ a_2 &= -3[\left(q^i\right)^{\text{nc}}(\tau_0^k) + \nu V^{1/2}(\tau_0^k)] \\ a_0 &= \left(q^i\right)^{\text{nc}}(\tau_0^k) \end{aligned} \tag{8}$$

- all the other components of $q_d^*(\cdot)$ are frozen:

$$(q_d^*)_{n-p}(t) = q_{n-p}^{\text{nc}}(\tau_0^k), \quad t \in (\tau_0^k, t_f^k] \tag{9}$$

As we said before, behind the choice of $q_d^*(\cdot)$ is the strategy to assure a robust stabilization on $\partial\Phi$ by mimicking the bouncing-ball dynamics. On the other hand this enables one to compute suitable upper-bounds that will help using Proposition 1.

In order to limit the deformation of the desired trajectory $q_d^*(\cdot)$ w.r.t. $q^{\text{nc}}(\cdot)$ during the I_k phases, we impose in the sequel

$$||q_p^{\text{nc}}(\tau_0^k)|| \leq \nu_1 \tag{10}$$

where $\nu_1 > 0$ is chosen by the designer. It is obvious that a smaller ν_1 leads to smaller deformation of the desired trajectory and to smaller deformation of the real trajectory as we shall see in Section 7. Nevertheless, due to the tracking error, ν_1 cannot be chosen zero. We also note that (10) is a practical way to choose τ_0^k.

During the transition phases I_k we define $(q_d)_{n-p}(t) = (q_d^*)_{n-p}(t)$. Assuming a finite accumulation period, the impact process can be considered in some way equivalent to a plastic impact. Therefore, $(q_d)_p(\cdot)$ and $(\dot{q}_d)_p(\cdot)$ are set to zero on the right of t_0^k. It is worth to recall that the first impact time t_0^k of each cycle k, is unknown.

4 Design of the desired contact force during constraint phases

The desired contact force $P_d = D^\top \lambda_d$ must be designed such that it is large enough to assure the constraint motion on the $\Omega_{2k+1}^{B_k}$-phases. Some contact force components have also to be decreased at the end of the $\Omega_{2k+1}^{B_k}$-phases in order to allow the detachment. Therefore we need a lower bound of the desired force which assures both the contact (without any undesired detachment which can generate other impacts) during the $\Omega_{2k+1}^{B_k}$ phases and a smooth detachment at the end of $\Omega_{2k+1}^{B_k}$. Dropping the time argument, the dynamics of the system on $\Omega_{2k+1}^{B_k}$ can be written as

$$\begin{cases} M(q)\ddot{q} + F = (1+K_f)D_p^\top(\lambda - \lambda_d) \\ J\dot{s}_2 + \gamma_1 s_2 + K(\tilde{\theta} - \tilde{q}) = 0 \\ 0 \leq q_p \perp \lambda_p \geq 0 \end{cases} \tag{11}$$

where $F = F(q, \dot{q}, \tilde{q}, \dot{\tilde{q}}, \tilde{\theta}) = -M(q)\ddot{q}_r + C(q, \dot{q})s_1 + \gamma_1 s_1 + K(\tilde{q} - \tilde{\theta})$ and $D_p = [I_p \vdots O_{p\times(n-p)}] \in \mathbb{R}^{p\times n}$. On $\Omega_{2k+1}^{B_k}$ the system has to be permanently constrained which is equivalent to $q_p(\cdot) = 0$ and $\dot{q}_p(\cdot) = 0$. In order to assure these conditions it is sufficient to have $\lambda_p > 0$.

We denote $M^{-1}(q) = \begin{pmatrix} [M^{-1}(q)]_{p,p} & [M^{-1}(q)]_{p,n-p} \\ [M^{-1}(q)]_{n-p,p} & [M^{-1}(q)]_{n-p,n-p} \end{pmatrix}$ and $C(q, \dot{q}) = \begin{pmatrix} C(q, \dot{q})_{p,p} & C(q, \dot{q})_{p,n-p} \\ C(q, \dot{q})_{n-p,p} & C(q, \dot{q})_{n-p,n-p} \end{pmatrix}$ where the meaning of each component is obvious. Let us also denote by K_p the matrix made of the first p rows and p columns of K.

Proposition 2 *On $\Omega_k^{B_k}$ the constraint motion of the closed-loop system (11),(4),(5) is assured if the desired contact force is defined by*

$$(\lambda_d)_p \triangleq \nu_p + \frac{K_p\tilde{\theta}_p}{1+K_f} - \frac{\bar{M}_{p,p}(q)}{1+K_f}\Big([M^{-1}(q)]_{p,p}C_{p,n-p}(q, \dot{q}) + [M^{-1}(q)]_{p,n-p}(C_{n-p,n-p}(q, \dot{q}) + \gamma_1 I_{n-p})\Big)(s_1)_{n-p} \tag{12}$$

where $\bar{M}_{p,p}(q) = \left([M^{-1}(q)]_{p,p}\right)^{-1} = \left(D_p M^{-1}(q) D_p^T\right)^{-1}$ is the inverse of the so-called Delassus' matrix Moreau (1988) and $\nu_p \in \mathbb{R}^p$, $\nu_p > 0$.

PROOF. It is noteworthy that the third relation in (11) implies on $\Omega_{2k+1}^{B_k}$ (see Glocker (2001))

$$0 \leq \ddot{q}_p \perp \lambda_p \geq 0 \Leftrightarrow 0 \leq D_p\ddot{q} \perp \lambda_p \geq 0. \tag{13}$$

From (11) one easily gets:

$$\ddot{q} = M^{-1}(q)\big[-F + (1+K_f)D_p^\top(\lambda - \lambda_d)_p\big]$$

Combining the last two equations we obtain the following

LCP with unknown λ:

$$0 \le D_p M^{-1}(q)\big[-F-(1+K_f)D_p^\top(\lambda_d)_p\big] +(1+K_f)D_pM^{-1}(q)D_p^\top\lambda_p \perp \lambda_p \ge 0 \tag{14}$$

Since $(1+K_f)D_pM^{-1}(q)D_p^\top > 0$ and hence is a P-matrix, the LCP (14) has a unique solution and one deduces that $\lambda_p > 0$ if and only if

$$\begin{aligned}
&\frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)\big[-F-(1+K_f)D_p^T(\lambda_d)_p\big] < 0 \Leftrightarrow\\
&(\lambda_d)_p > -\frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)F \Leftrightarrow\\
&(\lambda_d)_p = \nu_p - \frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)F
\end{aligned}\tag{15}$$

with $\nu_p \in \mathbb{R}^p$, $\nu_p > 0$. Since $F = -M(q)\ddot{q}_r + C(q,\dot{q})s_1 + \gamma_1 s_1 + K(\tilde{q}-\tilde{\theta})$, $(\ddot{q}_r)_p = 0$ and $(s_1)_p = 0$, (15) rewrites as (12) and the proof is finished. It is noteworthy that the solution of the LCP (14) is

$$\begin{aligned}
\lambda_p &= \frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)\big[F+(1+K_f)D_p^\top(\lambda_d)_p\big]\\
&= (\lambda_d)_p + \frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)F = \nu_p
\end{aligned}\tag{16}$$

where (12) has been used.

5 Strategy for take-off at the end of constraint phases $\Omega_{2k+1}^{B_k}$

In this Section we are interested in finding the conditions on the control signal $U_c^{B_k}$ that assures the take-off at the end of constraint phases $\Omega_{2k+1}^{B_k}$. As we have already seen before, the phase $\Omega_{2k+1}^{B_k}$ corresponds to the time interval $[t_f^k, t_d^k)$. The dynamics on $[t_f^k, t_d^k)$ is given by (11) and the system is permanently constrained, which implies $q_p(\cdot) = 0$ and $\dot{q_p}(\cdot) = 0$. Let us also consider that the first h constraints ($h < p$) have to be deactivated. Thus, the detachment takes place at t_d^k if $\ddot{q}_h(t_d^{k+}) > 0$ which requires $\lambda_h(t_d^{k-}) = 0$. The last $p-h$ constraints remain active which means $\lambda_{p-h}(t_d^{k-}) > 0$.

To simplify the notation we drop the time argument in many equations of this section. We decompose the LCP matrix (which is the Delassus' matrix multiplied by $1+K_f$) as:

$$(1+K_f)D_pM^{-1}(q)D_p^T = \begin{pmatrix} A_1(q) & A_2(q)\\ A_2(q)^T & A_3(q)\end{pmatrix} \tag{17}$$

with $A_1 \in \mathbb{R}^{h\times h}$, $A_2 \in \mathbb{R}^{h\times(p-h)}$ and $A_3 \in \mathbb{R}^{(p-h)\times(p-h)}$

Proposition 3 *The closed-loop system (11) (4) (5) is permanently constrained on $[t_f^k, t_d^k)$ and a smooth detachment is guaranteed on $[t_d^k, t_d^k+\epsilon)$ (ϵ is a small positive real number chosen by the designer) if*

(i) $$\begin{pmatrix} (\lambda_d)_h(t_d^k)\\ (\lambda_d)_{p-h}(t_d^k)\end{pmatrix} = \begin{pmatrix} \left(A_1 - A_2A_3^{-1}A_2^T\right)^{-1}\left(b_h - A_2A_3^{-1}b_{p-h}\right) - C_1(t-t_d^k)\\ C_2 + A_3^{-1}\left(b_{p-h} - A_2^T(\lambda_d)_h\right)\end{pmatrix} \tag{18}$$

where

$$b_p \triangleq b(q,\dot{q},U_c^{\emptyset}) \triangleq -D_pM^{-1}(q)F \ge 0$$

and $C_1 \in \mathbb{R}^h$, $C_2 \in \mathbb{R}^{p-h}$ such that $C_1 \ge 0$, $C_2 > 0$.

(ii) *On $[t_d^k, t_d^k+\epsilon)$*

$$q_d^*(t) = q_d(t) = \begin{pmatrix} q_h^*(t)\\ q_{n-h}^{nc}(t)\end{pmatrix},$$

where $q_h^*(\cdot)$ is a twice-differentiable function such that

$$\begin{aligned}
&q_h^*(t_d^k) = 0,\ q_h^*(t_d^k+\epsilon) = q_h^{nc}(t_d^k+\epsilon),\\
&\dot{q}_h^*(t_d^k) = 0,\ \dot{q}_h^*(t_d^k+\epsilon) = \dot{q}_h^{nc}(t_d^k+\epsilon)
\end{aligned}\tag{19}$$

and $\ddot{q}_h^*(t_d^{k+}) = a > \max\big(0, -A_1(q)(\lambda_d)_h(t_d^{k-})\big)$.

PROOF. See Appendix .1.

6 Closed-loop stability analysis

To simplify the notation $V(t, s(t), \tilde{\psi}(t))$ is denoted as $V(t)$. In order to introduce the main result of this paper we make the next assumption, which is verified in practice for dissipative systems with $e \in [0,1)$.

Assumption 1 *The controller U in (4) (5) assures that all the transition phases are finite.*

Lemma 1 *Consider the closed-loop system (1) (4) (5) with $(q_d^*)_p(\cdot)$ defined on the interval $[\tau_0^k, t_0^k]$ as in (7)-(9). Let us also suppose that condition* **b)** *of Proposition 1 is satisfied. The following inequalities hold:*

$$\begin{aligned}
&||\tilde{q}(t_0^{k-})|| \le \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}, \quad ||s_1(t_0^{k-})|| \le \sqrt{\frac{2V(\tau_0^k)}{\lambda_{min}(M(q))}},\\
&||\tilde{\theta}(t_0^{k-})|| \le \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}, \quad ||s_2(t_0^{k-})|| \le \sqrt{\frac{2V(\tau_0^k)}{\lambda_{min}(J)}},
\end{aligned}\tag{20}$$

and

$$
\begin{aligned}
||\dot{\tilde{q}}(t_0^{k-})|| &\leq \left(\sqrt{\frac{2}{\lambda_{min}(M(q))}} + \sqrt{\frac{\gamma_2}{\gamma_1}}\right) V^{1/2}(\tau_0^k) \\
||\dot{\tilde{\theta}}(t_0^{k-})|| &\leq \left(\sqrt{\frac{2}{\lambda_{min}(J)}} + \sqrt{\frac{\gamma_2}{\gamma_1}}\right) V^{1/2}(\tau_0^k)
\end{aligned}
\tag{21}
$$

Furthermore, if $t_0^k \leq \tau_1^k$ *one has*

$$
\begin{aligned}
&||(q_d)_p(t_0^{k-})|| \leq \epsilon + \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}, \\
&||(\dot{q}_d)_p(t_0^{k-})|| \leq \bar{k} + k^* V^{1/6}(\tau_0^k) \\
&||(\ddot{q}_d)_p(t_0^{k-})|| \leq 6\sqrt{2}\big(||q_p^{nc}(\tau_0^k)|| + \sqrt{p}\nu V^{1/2}(\tau_0^k)\big) \\
&||(q_d^{(3)})_p(t_0^{k-})|| \leq 6\sqrt{2}\big(||q_p^{nc}(\tau_0^k)|| + \sqrt{p}\nu V^{1/2}(\tau_0^k)\big)
\end{aligned}
\tag{22}
$$

where ϵ *is the real constant fixed in Definition 1 and* $\bar{k},\ k^* > 0$ *are some constant real numbers that will be defined in the proof.*

PROOF. See Appendix .2.

It is noteworthy that $q(\cdot)$ is a continuous signal. Nevertheless $\dot{q}(\cdot)$ presents discontinuities of the first kind at the impact times. From (5) one deduces that the controller U_r jumps also at the impact times generating a jump in the desired signal $\theta_d(\cdot)$. Therefore, in order to study the evolution of the Lyapunov function candidate (6) one has to analyze $\sigma_{\tilde{\theta}}(\cdot)$ and $\sigma_{\dot{\tilde{\theta}}}(\cdot)$.

Lemma 2 *The controller* U *in (4) (5) guarantees that* $||\sigma_{\tilde{\theta}}(\cdot)||,\ ||\sigma_{\dot{\tilde{\theta}}}(\cdot)|| \in S(1) \equiv L^\infty$.

PROOF. See Appendix .3.

We now state the main result of this paper.

Theorem 1 *Let Assumption 1 hold,* $e = 0$ *and* $q_d^*(\cdot)$ *defined as in (7)-(9). The closed-loop system (1) (4) (5) initialized on* Ω_0 *such that* $V(\tau_0^0) \leq 1$*, satisfies the requirements of Proposition 1 and is therefore practically weakly stable with the closed-loop state* $x(\cdot) = [\tilde{\psi}(\cdot), s(\cdot)]$ *and* $R = \sqrt{e^{-\gamma(t_f^k - t_\infty^k)}(\rho^* + \xi)/\bar{\rho}}$ *where* ρ^*, $\bar{\rho}$ *and* ξ *are defined in the proof.*

PROOF. See Appendix .4.

7 Illustrative example

Some experimental results are obtained by simulating the behavior of a planar two-link flexible-joint manipulator in presence of two constraints. As in Morărescu and Brogliato (2008) we impose an admissible domain $\Phi = \{(x, y) \mid y \geq 0,\ 0.7 - x \geq 0\}$. Let us also consider an unconstrained desired trajectory $q^{nc}(\cdot)$ whose orbit is given by the circle $\{(x, y) \mid (x-0.7)^2 + y^2 = 0.5\}$. It violates both constraints. In other words, the two-link planar manipulator must track a quarter-circle; stabilize on and then follow the line $\Sigma_1 = \{(x, y) \mid y = 0\}$; stabilize on the intersection of Σ_1 and $\Sigma_2 = \{(x, y) \mid x = 0.7\}$; detach from Σ_1 and follow Σ_2 until the unconstrained circle re-enters Φ and finally take-off from Σ_2 in order to repeat the previous steps. The task representation here is given by (see (2)) $B_{2k} = \emptyset, m_{2k} = 1,\ B_{2k,1} = \{1\}, B_{2k+1} = \{1\},\ m_{2k+1} = 2,\ B_{2k+1,1} = \{1, 2\},\ B_{2k+1,2} = \{2\}$. The numerical values used for the dynamical model are $l_1 = l_2 = 0.5m$, $m_1 = m_2 = 1kg$, $I_1 = I_2 = 0.5kg.m^2$, $J_1 = J_2 = 0.1kg.m^2$ and the impacts are imposed by $\nu = 10$ in (7) (8). The stiffness matrix is defined by $K = diag(2000N/m, 2000N/m)$. Let us say that the quarter-circle is completely tracked in one round. We set the period of each round to 10 seconds and we simulate the dynamics during 6 rounds using the Moreau's time-stepping algorithm of the SICONOS software platform (Acary and Brogliato (2008)). We set the controller gains $\gamma_1 = 10$, $\gamma_2 = 1$ and we choose $\nu_1 = 0.1$ (like this we implicitly set τ_0^k see (10)) in order to better point out the deformation of $q_d(\cdot)$ on the transition phases (Figure 1 (left)). In Figure 2 we have shifted backward the desired trajectory on $I_2^{B_2}$ to highlight that the Lyapunov function at the instant τ_0^k is smaller when k increases.

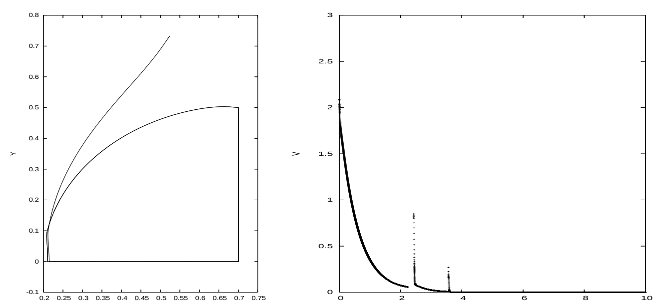

Fig. 1. Left:The trajectory of the system during 6 rounds; Right: The variation of the almost non-increasing Lyapunov function during the first round.

The behavior of the system during one round is emphasized in Figure 1 (right) and the shape of the control law is depicted in Figure 3.

7.1 Compensation of flexibilities

As noticed in Brogliato *et al* (2007) the control laws designed for rigid systems (the Slotine & Li control and its

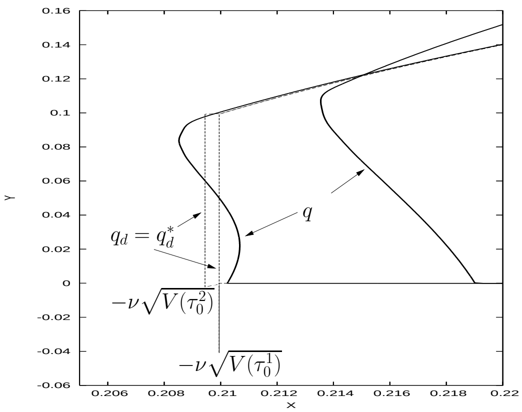

Fig. 2. Zoom on the transition phases $I_{2k}^{B_{2k}}$.

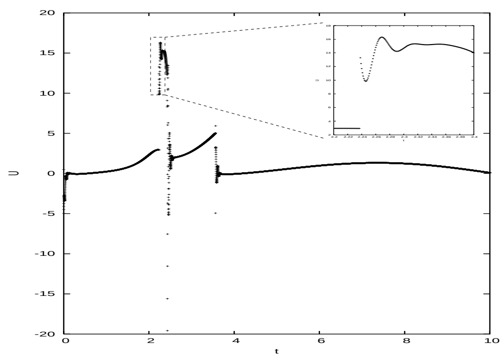

Fig. 3. The control law applied to θ_1 during the first round.

adaptation for systems with one or multiple constraints Bourgeot and Brogliato (2005); Morărescu and Brogliato (2008)) behave well for manipulators with large joint stiffness (see also Figure 4 for the multi-constraint case).

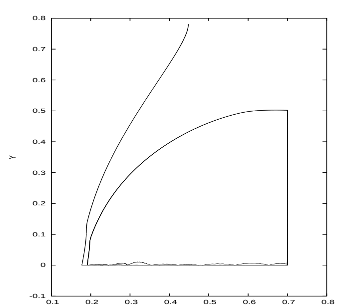

Fig. 4. The variation of the end-effector coordinates using the rigid controller when the stiffness matrix is defined by $K = diag(5000N/m, 5000N/m)$.

In order to highlight the importance of flexibilities' compensation we keep the numerical values used in the previous Subsection with one exception, the stiffness matrix is defined by $K = diag(200N/m, 200N/m)$. Using the control with no flexibility compensation (named the "rigid controller") one obtains a completely deteriorated behavior (see Figure 5). Furthermore, the control signal oscillates very much after the first impact (Figure 6).

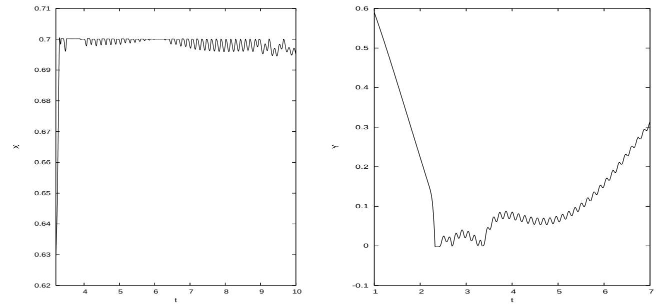

Fig. 5. The variation of the end-effector coordinates using the rigid controller

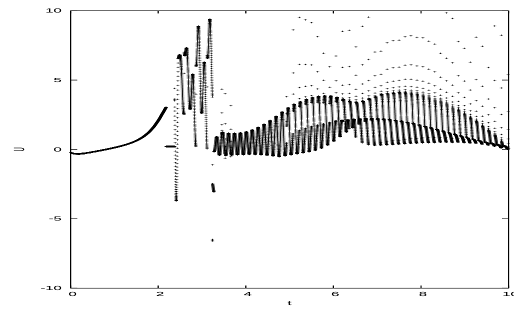

Fig. 6. The rigid control applied to θ_1 during the first round.

On the other hand using the controller designed in this paper the desired trajectory is well tracked (see Figure 7) and the control signal is quickly stabilized during the I_k phases (see Figure 8).

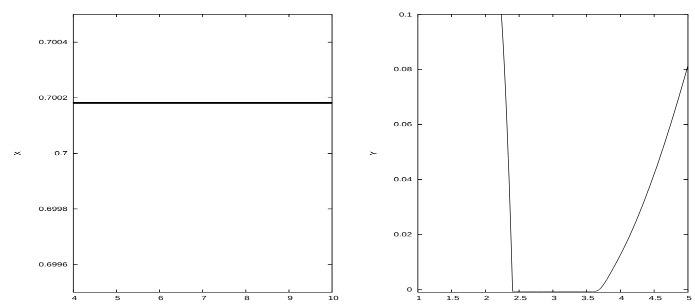

Fig. 7. The variation of the end-effector coordinates using with the controller (4) (5).

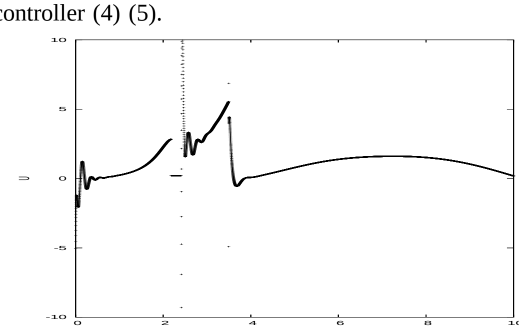

Fig. 8. The control law applied to θ_1 during the first round.

More numerical results can be found in Morărescu *et al* (2008)

8 Conclusions

In this paper we have proposed a solution for the control of nonsmooth Lagrangian systems with flexible-joint. All the ingredients entering the dynamics (desired trajectories, desired contact forces, exogenous instants playing a role in the definition of the control law) are explicitly defined. Numerical simulations are done with the SICONOS software platform in order to illustrate the results.

.1 *Proof of Proposition 3*

The necessary condition for take-off after the instant t_d^k is given by $\lambda_h(t_d^{k-}) = 0$ and $\lambda_{p-h}(t_d^{k-}) > 0$. Precisely, we impose a positive contact force on $[t_f^k, t_d^k)$ with the first h components approaching 0 when t approaches t_d^k. From (17) and (11) it is straightforward that the LCP (13) rewrites as:

$$0 \le \begin{pmatrix} \lambda_h \\ \lambda_{p-h} \end{pmatrix} \perp \begin{pmatrix} b_h + A_1(\lambda-\lambda_d)_h + A_2(\lambda-\lambda_d)_{p-h} \\ b_{p-h} + A_2^T(\lambda-\lambda_d)_h + A_3(\lambda-\lambda_d)_{p-h} \end{pmatrix} \ge 0 \tag{.1}$$

Since $(1+K_f)D_pM^{-1}(q)D_p^T > 0$, the LCP (13) (or the equivalent one (.1)) has a unique solution. Imposing $\lambda_h = 0$ one gets

$$0 \le \lambda_{p-h} \perp b_{p-h} - A_2^T(\lambda_d)_r + A_3(\lambda-\lambda_d)_{p-h} \ge 0$$

with the solution

$$\lambda_{p-h} = -A_3^{-1}\left(b_{p-h} - A_2^T(\lambda_d)_h - A_3(\lambda_d)_{p-h}\right) \tag{.2}$$

Thus $\lambda_{p-h} > 0$ is equivalent to

$$(\lambda_d)_{p-h} > A_3^{-1}\left(b_{p-h} - A_2^T\,(\lambda_d)_h\right)$$

which leads to the second part of definition (18). Furthermore, replacing $(\lambda_d)_{p-h}$ in (.2) we get $\lambda_{p-h} = C_2$ and $b_h + A_1(\lambda-\lambda_d)_h + A_2(\lambda-\lambda_d)_{p-h} \ge 0$ yields the first part of definition (18). Consequently the solution of the LCP (.1) is $\lambda_p = \begin{pmatrix} 0 \\ C_2 \end{pmatrix} \in \mathbb{R}^p$ when $(\lambda_d)_p$ is defined by (18).

The jumps in the Lyapunov function are avoided during the detachment phase using a twice differentiable desired trajectory $q_d(\cdot)$ defined as in item **(ii)** of the Proposition. In order to assure a smooth detachment (without impacts) on $[t_d^k, t_d^k+\epsilon)$ we need a large enough positive desired acceleration $(\ddot{q}_d)_h$. At t_d^{k-} one has

$$\ddot{q}_h(t_d^{k-}) = -D_hM^{-1}(q)\left[F + (1+K_f)D_h^\top(\lambda_d)_h\right]$$

while at t_d^{k+} one has $\ddot{q}_{p-h}(t_d^{k+}) = D_hM^{-1}(q)F$. Since $(\ddot{q}_d)_h(t_d^{k-}) = 0$ we arrive at

$$\sigma_{\ddot{q}_h(t_d^k)} = (\ddot{q}_d)_h(t_d^{k+}) + A_1(q)(\lambda_d)_h(t_d^{k-})$$

Therefore $\ddot{q}_{1d}(t_d^{k+})$ has to be positive and large enough in order to compensate for $-A_1(q)(\lambda_d)_h(t_d^{k-})$ at the instant t_d^k. Consequently one defines $\ddot{q}_1^*(t_d^{k+}) = a > \max\left(0, -A_1(q)(\lambda_d)_h(t_d^{k-})\right)$ and the detachment is assured.

.2 *Proof of Lemma 1*

From (6) we can deduce that $V(t_0^{k-}) \ge \gamma_1\gamma_2||\tilde{q}(t_0^{k-})||^2$, $V(t_0^{k-}) \ge \frac{1}{2}s_1^\top(t_0^{k-})M(q(t_0^{k-}))s_1(t_0^{k-})$ and

$$V(t_0^{k-}) \ge \gamma_1\gamma_2||\tilde{\theta}(t_0^{k-})||^2, \quad V(t_0^{k-}) \ge \frac{1}{2}s_2^\top(t_0^{k-})Js_2(t_0^{k-})$$

Since condition **b)** of Proposition 1 is satisfied one has $V(\tau_0^k) \ge V(t_0^{k-})$ and (20) becomes trivial. Let us recall that $s_1(t) = \dot{\tilde{q}}(t) + \gamma_2\tilde{q}(t)$ and $s_2(t) = \dot{\tilde{\theta}}(t) + \gamma_2\tilde{\theta}(t)$ which implies $||\dot{\tilde{q}}(t_0^{k-})|| \le ||s_1(t_0^{k-})|| + \gamma_2||\tilde{q}(t_0^{k-})||$ and $||\dot{\tilde{\theta}}(t_0^{k-})|| \le ||s_2(t_0^{k-})|| + \gamma_2||\tilde{\theta}(t_0^{k-})||$ respectively. Combining this with (20) we derive (21).

The proof of (22) follows the ideas presented in Morărescu and Brogliato (2008). Roughly the first inequality in (22) is based on the definition of p_ϵ-impacts (see Definition 1). The remaining inequalities in (22) are based on the particular definition of $(q_d^*)_p(\cdot)$ (see (7) (8)). The upper bound of $||(\dot{q}_d)_p(t_0^{k-})||$ was derived in Morărescu and Brogliato (2008) with $\bar{k} = \dfrac{6\sqrt{p\nu_1\epsilon}}{\tau_1^k - \tau_0^k}$ and

$$k^* = \frac{6\sqrt{p}}{\tau_1^k - \tau_0^k}\sqrt{\left(\frac{1}{\sqrt{\gamma_1\gamma_2}} + \nu\right)(\nu+\nu_1) + \epsilon\nu}$$

Finally, differentiating (7) two and three times respectively one obtains

$$\begin{aligned} \ddot{q}_d^i(t_0^{k-}) &= \lim_{t\to t_0^k, t<t_0^k} 6((q^i)^{\text{nc}}(\tau_0^k) + \nu V^{1/2}(\tau_0^k))(2t'-1) \\ &\le \lim_{t\to t_0^k, t<t_0^k} 6((q^i)^{\text{nc}}(\tau_0^k) + \nu V^{1/2}(\tau_0^k)) \\ (q_d^i)^{(3)}(t_0^{k-}) &= \lim_{t\to t_0^k, t<t_0^k} 6((q^i)^{\text{nc}}(\tau_0^k) + \nu V^{1/2}(\tau_0^k)) \end{aligned}$$

which leads to the upper bounds of $||(\ddot{q}_d)_p(t_0^{k-})||$ and $||(q_d^{(3)})_p(t_0^{k-})||$ respectively.

.3 Proof of Lemma 2

Since $\theta(\cdot), \dot{\theta}(\cdot)$ are continuous on $\mathbb{R}_+$ and $\theta_d(\cdot), \dot{\theta}_d(\cdot)$ are continuous on $\mathbb{R}_+ \setminus \{t_0^k \mid k \in \mathbb{Z}\}$ one deduces that $\sigma_{\tilde{\theta}}(t) = 0 = \sigma_{\dot{\tilde{\theta}}}(t)$, $\forall t \neq t_0^k$. Therefore Lemma 2 holds if there exist some real constants that upper bound $||\sigma_{\tilde{\theta}}(t_0^k)||$, $||\sigma_{\dot{\tilde{\theta}}}(t_0^k)||$, $\forall k \in \mathbb{Z}$. The definition of $\theta_d(\cdot)$ (see (4)) allows us to write

$$\begin{aligned}
\sigma_{\tilde{\theta}}(t_0^k) &= -\sigma_{\theta_d}(t_0^k) = -\sigma_{q_d}(t_0^k) - K^{-1}\sigma_{U_r}(t_0^k) \\
&= \begin{pmatrix} (q_d)_p(t_0^{k-}) \\ 0 \end{pmatrix} - K^{-1}\sigma_{U_r}(t_0^k) \\
\sigma_{\dot{\tilde{\theta}}}(t_0^k) &= -\sigma_{\dot{\theta}_d}(t_0^k) = -\sigma_{\dot{q}_d}(t_0^k) - K^{-1}\sigma_{\dot{U}_r}(t_0^k) \\
&= \begin{pmatrix} (\dot{q}_d)_p(t_0^{k-}) \\ 0 \end{pmatrix} - K^{-1}\sigma_{\dot{U}_r}(t_0^k)
\end{aligned} \tag{.3}$$

Therefore

$$\begin{aligned}
||\sigma_{\tilde{\theta}}(t_0^k)|| &\leq ||(q_d)_p(t_0^{k-})|| + \lambda_{max}(K^{-1})||\sigma_{U_r}(t_0^k)|| \\
||\sigma_{\dot{\tilde{\theta}}}(t_0^k)|| &\leq ||(\dot{q}_d)_p(t_0^{k-})|| + \lambda_{max}(K^{-1})||\sigma_{\dot{U}_r}(t_0^k)||
\end{aligned}$$

Using (5) one obtains

$$\sigma_{U_r}(t_0^k) = M(q)\sigma_{\ddot{q}_r}(t_0^k) + \sigma_{C(q,\dot{q})\dot{q}_r}(t_0^k) - \gamma_1\sigma_{s_1}(t_0^k)$$

From (9) one has $(\dot{q}_d)_{n-p}(t) = 0$, $(\ddot{q}_d)_{n-p}(t) = 0$ $\forall t \in [\tau_0^k, t_f^k]$. Moreover, as we have mentioned at the end of Section 3, $(q_d)_p(\cdot)$, $(\dot{q}_d)_p(\cdot)$ and implicitly $(\ddot{q}_d)_p(\cdot)$ are set to zero on $(t_0^k, t_f^k]$. Thus taking into account the relation $||\dot{q}(t_0^{k+})|| \leq \mathbf{w}||\dot{q}(t_0^{k-})||$ (where $\mathbf{w} = \sqrt{\frac{\lambda_{max}(M)}{\lambda_{min}(M)}}$) and Property 1 one arrives at

$$\begin{aligned}
&||\sigma_{\ddot{q}_r}(t_0^k)|| \leq ||(\ddot{q}_d)_p(t_0^{k-})|| + \gamma_2||(\dot{q}_d)_p(t_0^{k-})|| \\
&\qquad\qquad + \gamma_2(1+\mathbf{w})||\dot{q}(t_0^{k-})|| \\
&||\sigma_{C(q,\dot{q})\dot{q}_r}(t_0^k)|| \leq ||\sigma_{C(q,\dot{q})}\dot{q}_r(t_0^{k-})|| + ||C(q,\dot{q}(t_0^{k+}))\sigma_{\dot{q}_r}(t_0^k)|| \\
&\in S\big(2(1+\gamma_2)||\dot{q}(t_0^{k-})||||(\dot{q}_d)_p(t_0^{k-})|| + \gamma_2||(q_d)_p(t_0^{k-})||\big) \\
&||\sigma_{s_1}(t_0^k)|| \leq (1+\mathbf{w})||\dot{q}(t_0^{k-})|| + ||(\dot{q}_d)_p(t_0^{k-})|| \\
&\qquad\qquad + \gamma_2||(q_d)_p(t_0^{k-})||
\end{aligned} \tag{.4}$$

When $V(\tau_0^k) \leq 1$, Lemma 1 states that $||(\dot{q}_d)_p(t_0^{k-})||$, $||(q_d)_p(t_0^{k-})||$ and $||\dot{q}(t_0^{k-})||$ are bounded by some constants. Thus all the quantities in (.4) are bounded by some constants independent of the cycle index k. This means that $||\sigma_{U_r}(t_0^k)||$ is bounded by a constant independent of the cycle index, which implies the same for $||\sigma_{\tilde{\theta}}(t_0^k)||$. In other words $||\sigma_{\tilde{\theta}}(t)|| \in S(1)$.

Differentiating (5) one obtains

$$\begin{aligned}
\dot{U}_r(t) &= M(q)q_r^{(3)}(t) + \dot{M}(q)\ddot{q}_r(t) + C(q,\dot{q})\ddot{q}_r(t) \\
&\quad + \dot{C}(q,\dot{q})\dot{q}_r(t) + \frac{\partial G}{\partial q}\dot{q}(t) - \gamma_1\dot{s}_1(t)
\end{aligned} \tag{.5}$$

where $\dot{M}$, $\dot{C}$ stand for $\frac{dM}{dt}$ and $\frac{dC}{dt}$ respectively.
It is clear that

$$\dot{C}(q,\dot{q})(t) = \frac{\partial C}{\partial q}(q,\dot{q})\dot{q}(t) + \frac{\partial C}{\partial \dot{q}}(q,\dot{q})\ddot{q}(t)$$

and using Properties 1 and 3 one derives

$$||\dot{C}(q,\dot{q})(t)|| \in S(||\dot{q}(t)||^2 + ||\ddot{q}(t)||)$$

Furthermore, Lemma 1 and the first equation in (1) assure that $||\dot{q}(t)||^2, ||\ddot{q}(t)|| \in S(1)$. Thus $||\dot{C}(q,\dot{q})(\cdot)||$, $||\sigma_{\dot{C}(q,\dot{q})}(\cdot)|| \in S(1)$ and one derives that

$$\begin{aligned}
||\sigma_{\dot{C}(q,\dot{q})}(t_0^k)\dot{q}_r(t_0^k)|| &\leq ||\sigma_{\dot{C}(q,\dot{q})}(t_0^k)||.||\dot{q}_r(t_0^{k+})|| \\
&\quad + ||\dot{C}(q,\dot{q})(t_0^{k-})||.||\sigma_{\dot{q}_r}(t_0^k)|| \in S(1)
\end{aligned} \tag{.6}$$

Property 1 allows us to replace $\dot{M}(q)$ by $C(q,\dot{q}) + C^\top(q,\dot{q})$ which leads to

$$\begin{aligned}
&\dot{M}(q)\ddot{q}_r(t) + C(q,\dot{q})\ddot{q}_r(t) = (2C(q,\dot{q}) + C^\top(q,\dot{q}))\ddot{q}_r(t) \Rightarrow \\
&||\dot{M}(q)\ddot{q}_r(t) + C(q,\dot{q})\ddot{q}_r(t)|| \leq 3||C(q,\dot{q})||.||\ddot{q}_r(t)|| \Rightarrow \\
&||\dot{M}(q)\ddot{q}_r(t) + C(q,\dot{q})\ddot{q}_r(t)|| \in S(||\dot{q}||.||\ddot{q}_r(t))||)
\end{aligned}$$

Since $||\ddot{q}_r(t))|| \leq ||\ddot{q}_d(t)|| + \gamma_2||\dot{\tilde{q}}(t)||$, using Lemma 1 one gets

$$||\dot{M}(q)\ddot{q}_r(t) + C(q,\dot{q})\ddot{q}_r(t)|| \in S(1) \tag{.7}$$

The definitions (7)-(9) and the first equation in (1) assure that $||q_r^{(3)}(t)|| \in S(1)$. Therefore

$$||M(q)q_r^{(3)}(t)|| \leq \lambda_{max}(M)||q_r^{(3)}(t)|| \in S(1) \tag{.8}$$

Property 2 states that $||\frac{\partial G}{\partial q}|| \in S(1)$, which implies

$$\left.\begin{aligned} &\left|\left|\tfrac{\partial G}{\partial q}\dot{q}(t)\right|\right| \in S(||\dot{q}(t)||) \\ &||\dot{q}(t)|| \in S(1) \end{aligned}\right\} \Rightarrow \left|\left|\frac{\partial G}{\partial q}\dot{q}(t)\right|\right| \in S(1) \tag{.9}$$

Introducing (.6)–(.9) in (.5) and taking into account the last inequality in (.4) we arrive at $||\sigma_{\dot{U}_r}(t)|| \in S(1)$ and thus $||\sigma_{\dot{\tilde{\theta}}}(t)|| \in S(1)$.

.4 Proof of Theorem 1

First we observe that conditions **a)** and **c)** of Proposition 1 hold when the hypothesis of the Theorem are verified. Thus

Theorem 1 holds if the conditions **b)**, **d)** of Proposition 1 are verified.
b) Using that $\dot{M}(q) - 2C(q,\dot{q})$ is a skew-symmetric matrix (see Property 1), straightforward computations show that on $\mathbb{R}_+ \setminus \bigcup_{k\geq 0}[t_0^k, t_f^k]$ the time derivative of the Lyapunov function is given by

$$\begin{aligned}\dot{V}(t) =& -\gamma_1||\dot{\tilde{q}}||^2 - \gamma_1\gamma_2^2||\tilde{q}||^2 - \gamma_1||\dot{\tilde{\theta}}||^2 - \gamma_1\gamma_2^2||\tilde{\theta}||^2 \\ & -\gamma_2(\tilde{q}-\tilde{\theta})^\top K(\tilde{q}-\tilde{\theta}) + (1+K_f)s_1^\top D_p^\top(\lambda-\lambda_d)_p \\ =& -\gamma_1||\dot{\tilde{q}}||^2 - \gamma_1\gamma_2^2||\tilde{q}||^2 - \gamma_1||\dot{\tilde{\theta}}||^2 - \gamma_1\gamma_2^2||\tilde{\theta}||^2 \\ & -\gamma_2(\tilde{q}-\tilde{\theta})^\top K(\tilde{q}-\tilde{\theta}) \leq 0\end{aligned}$$

where we have used the fact that $(q_d)_p \equiv 0$, $(\dot{q}_d)_p \equiv 0$, $q_p \equiv 0$, $\dot{q}_p \equiv 0$, thus $(s_1)_p \equiv 0$ on constraint phases and $\lambda_p \equiv 0$, $(\lambda_d)_p \equiv 0$ on free-motion phases. On the other hand

$$\begin{aligned}V(t) \leq& \frac{\lambda_{max}(M(q))}{2}||s_1||^2 + \frac{\lambda_{max}(J)}{2}||s_2||^2 + \gamma_1\gamma_2||\tilde{q}||^2 \\ & + \gamma_1\gamma_2||\tilde{\theta}||^2 + \frac{1}{2}(\tilde{q}-\tilde{\theta})^\top K(\tilde{q}-\tilde{\theta}) \\ \leq& \gamma^{-1}[\gamma_1||\dot{\tilde{q}}||^2 + \gamma_1\gamma_2^2||\tilde{q}||^2 + \gamma_1||\dot{\tilde{\theta}}||^2 + \gamma_1\gamma_2^2||\tilde{\theta}||^2 \\ & + \gamma_2(\tilde{q}-\tilde{\theta})^\top K(\tilde{q}-\tilde{\theta})]\end{aligned}$$

where

$$\begin{aligned}\gamma^{-1} = \max\Big\{&\lambda_{max}(\mathbf{M}(q))\frac{1+2\gamma_2}{2\gamma_1}; \\ & \frac{\lambda_{max}(\mathbf{M}(q))(\gamma_2+2)+2\gamma_1}{2\gamma_1\gamma_2}; \frac{1}{2\gamma_2}\Big\} > 0\end{aligned}$$

with $\mathbf{M}(q) = \begin{pmatrix} M(q) & 0_{n\times n} \\ 0_{n\times n} & J \end{pmatrix}$. Therefore $\dot{V}(t) \leq -\gamma^{-1}V(t)$ on $\mathbb{R}_+ \setminus \bigcup_{k\geq 0}[t_0^k, t_f^k]$.

d) There is only one impact during each transition phase since $e = 0$ and with the choice of U_t^B in (5). Therefore $V(t_\infty^k) = V(t_0^{k-}) + \sigma_V(t_0^k) \leq V(\tau_0^k) + \sigma_V(t_0^k)$. We compute now the jump of the Lyapunov function at the impact time t_0^k. Let $\mathcal{K} = \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}$ and $\psi = (q^\top, \theta^\top)^\top$.

$$\begin{aligned}& V(t_0^{k+}) - V(t_0^{k-}) = \gamma_1\gamma_2\sigma_{\tilde{\psi}^\top\tilde{\psi}}(t_0^k) + \\ & \frac{1}{2}\left(s^\top(t_0^{k+})\mathbf{M}(q)s(t_0^{k+}) - s^\top(t_0^{k-})\mathbf{M}(q)s(t_0^{k-})\right) + \\ & \frac{1}{2}\left(\tilde{\psi}^{\top}(t_0^{k+})\mathcal{K}\tilde{\psi}(t_0^{k+}) - \tilde{\psi}^{\top}(t_0^{k-})\mathcal{K}\tilde{\psi}(t_0^{k-})\right) \qquad (.10)\end{aligned}$$

Replacing $\tilde{\psi}(t_0^{k+})$ by $\tilde{\psi}(t_0^{k-}) + \sigma_{\tilde{\psi}}(t_0^k)$, the second term of the right hand side of (.10) becomes

$$\frac{1}{2}\left(2\tilde{\psi}^{\top}(t_0^{k-})\mathcal{K}\sigma_{\tilde{\psi}}(t_0^k) + \sigma_{\tilde{\psi}}^\top(t_0^k)\mathcal{K}\sigma_{\tilde{\psi}}(t_0^k)\right)$$

which is upper bounded by

$$\lambda_{max}(\mathcal{K})(||\tilde{\psi}(t_0^{k-})||\cdot||\sigma_{\tilde{\psi}}(t_0^k)|| + \frac{1}{2}||\sigma_{\tilde{\psi}}(t_0^k)||^2)$$

Therefore Lemma 1 and 2 imply that there exists a real positive constant c_1 such that

$$\frac{1}{2}\left(\tilde{\psi}^{\top}(t_0^{k+})\mathcal{K}\tilde{\psi}(t_0^{k+}) - \tilde{\psi}^{\top}(t_0^{k-})\mathcal{K}\tilde{\psi}(t_0^{k-})\right) \leq c_1, \forall k \geq 0 \qquad (.11)$$

On the other hand

$$\begin{aligned}s^\top(t_0^{k+})\mathbf{M}(q)s(t_0^{k+}) - s^\top(t_0^{k-})\mathbf{M}(q)s(t_0^{k-}) = \\ \sigma_{s_1^\top M(q)s_1}(t_0^k) + \sigma_{s_2^\top Js_2}(t_0^k)\end{aligned}$$

It is easy to see that

$$\sigma_{s_2^\top Js_2}(t_0^k) = 2s_2^\top(t_0^{k-})J\sigma_{s_2}(t_0^k) + \sigma_{s_2}^\top(t_0^k)J\sigma_{s_2}(t_0^k)$$

and using Lemma 1, Lemma 2 and the relation $\sigma_{s_2}(t_0^k) = \sigma_{\dot{\tilde{\theta}}}(t_0^k) + \gamma_2\sigma_{\tilde{\theta}}(t_0^k)$ one deduces that there exist a real positive constant c_2 such that

$$\sigma_{s_2^\top Js_2}(t_0^k) \leq c_2, \quad \forall k \geq 0 \qquad (.12)$$

As proved in Morărescu and Brogliato (2008) there exists a real positive constant c_3 such that

$$\sigma_{s_1^\top M(q)s_1}(t_0^k) + \gamma_1\gamma_2\sigma_{\tilde{q}^\top\tilde{q}}(t_0^k) \leq c_3, \quad \forall k \geq 0 \qquad (.13)$$

Finally, Lemma 2 assures the existence of $c_4 \in \mathbb{R}_+$ such that

$$\gamma_1\gamma_2\sigma_{\tilde{\theta}^\top\tilde{\theta}}(t_0^k) \leq c_4, \quad \forall k \geq 0 \qquad (.14)$$

In conclusion, inserting (.11), (.12), (.13) and (.14) in (.10) one gets

$$V(t_0^{k+}) - V(t_0^{k-}) \leq c_1 + c_2 + c_3 + c_4, \quad \forall k \geq 0 \quad (.15)$$

Thus condition **d)** of Proposition 1 is verified for $\rho^* = 1$, $\xi = c_1 + c_2 + c_3 + c_4$ and the closed-loop system (1) (4) (5) is practically weakly stable with $R = \alpha^{-1}\left(e^{-\gamma(t_f^k - t_\infty^k)}(1+\xi)\right)$.
Let us consider $\bar{\rho} = \min\{\lambda_{min}(\mathbf{M}(q))/2; \gamma_1\gamma_2\}$. Defining $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$, $\alpha(\omega) = \bar{\rho}\omega^2$ we get $\alpha(0) = 0$, $\alpha(|||[s(t), \tilde{q}(t)]|||) \leq V(t, s, \tilde{q})$ and the proof is finished.

References

Acary, V. & Brogliato, B. (2008): *Numerical Methods for Nonsmooth Dynamical Systems*, Lecture Notes in Applied and Computational Mechanics, vol **35**, Springer-Verlag, Berlin Heidelberg.

Albu-Schaffer, A., Ott, C., Hirzinger,G. (2004): *A passivity based Cartesian impedance controller for flexible joint robots - part II: full state feedback, impedance design and experiments*, Proceedings ICRA '04, IEEE International Conference on Robotics and Automation, vol.3, pp.2666 - 2672, April 26-May 1.

Bentsman, J. & Miller, B. (2007b): *Dynamical systems with active singularities of elastic type: A modeling and controller synthesis framework*, IEEE Transactions on Automatic Control, vol.52, no 1, pp.39-55, January.

Bourgeot, J.-M.& Brogliato, B. (2005):*Tracking control of complementarity lagrangian systems*, International Journal of Bifurcation and Chaos, **15**(6), 1839-1866.

Brogliato, B. (1999): *Nonsmooth Mechanics*, Springer CCES, London, (1999), 2nd Ed.

Brogliato, B., Lozano, R., Maschke, B. and Egeland, O.(2007): *Dissipative Systems Analysis and Control. Theory and Applications*, Springer CCES, London, (2007), 2nd Ed.

Brogliato, B., Niculescu S.-I., & Orhant, P. (1997): *On the control of finite-dimensional mechanical systems with unilateral constraints*, IEEE Trans. Autom. Contr. **42**(2), 200-215.

Brogliato, B., Ortega, R., & Lozano, R.(1995): *Global tracking controllers for flexible-joint manipulators: a comparative study*, Automatica, **31**(7), 941-956.

Casagrande, D., Astolfi, A., & Parisini, T.: *Global asymptotic stabilization of the attitude and the angular rates of an underactuated non-symmetric rigid body*, Automatica, vol.44, no 7, pp.1781-1789 (2008).

Dzonou, R., & Monteiro Marques, M.D.P.(2007): *A sweeping process approach to inelastic contact problems with general inertia operators*, European Journal of Mechanics - A/Solids, **26**(3), 474-490.

Facchinei, F., & Pang, J.S. (2003): *Finite-dimensional Variational Inequalities and Complementarity Problems*, Springer Series in Operations Research, New-York.

Galeani, S., Menini, L., Potini, A. & Tornambè, A.(2008): *Trajectory tracking for a particle in elliptical billiards*, International Journal of Control, vol. **81**(2), 189-213.

Glocker, C. (2001): *Set Value Force Laws: Dynamics of Non-Smooth Systems* Lecture Notes in Applied Mechanics, Vol.1, Springer.

Lee, E., Park, J., Loparo, K.A., Schrader, C.B.& Chang, P.H. (2003): *Bang-bang impact control using hybrid impedance/time-delay control*, IEEE/AMSE Transactions on Mechatronics, vol.**8**, No.2, 272-277.

Leine, R.I. & van de Wouw, N. (2008a): *Stability properties of equilibrium sets of nonlinear mechanical systems with dry friction and impact*, Nonlinear Dynamics, Vol **51**(4), 551-583.

Leine, R.I. and van de Wouw, N. (2008b): *Uniform convergence of monotone measure differential inclusions: with application to the control of mechanical systems with unilateral constraints*, Int. Journal of Bifurcation and Chaos, vol.18, no 5, pp.1435-1457.

Mabrouk, M. (1998): *A unified variational model for the dynamics of perfect unilateral constraints*, European Journal of Mechanics A/Solids, **17**(5), 819-842.

McClamroch, N. & Wang, D. (1988): *Feedback stabilization and tracking of constrained robots*, IEEE Trans. Autom. Contr. **33**, 419-426.

Menini, L. & Tornambè, A. (2001): *Asymptotic tracking of periodic trajectories for a simple mechanical system subject to nonsmooth impacts*, IEEE Trans. Autom. Contr. **46**, 1122-1126.

Menini, L. & Tornambè, A. (2001): *Exponential and BIBS stabilisation of one degree of freedom mechanical system subject to single non-smooth impacts*, IEE Proc.- Control Theory Appl. Vol.**148**, No. 2, 147-155.

Miller, B. & Bentsman, J. (2006): *Optimal control problems in hybrid systems with active singularities*, Nonlinear Anal., Theory Methods Appl. 65, No. 5 (A), 999-1017.

Morărescu, I.-C. & Brogliato, B. (2008): *Passivity-based tracking control of multiconstraint complementarity Lagrangian systems*, IEEE Conference on Decision and Control, Cancun, Mexico, December 9-11. See also http://hal.inria.fr/docs/00/27/62/64/PDF/RR-6483.pdf

Morărescu, I.-C., Brogliato, B and Haad, S.: *Passivity-based switching control of flexible-joint complementarity mechanical systems*, http://hal.inria.fr/

Moreau, J.J.(1988): *Unilateral contact and dry friction in finite freedom dynamics*, Nonsmooth Mechanics and Applications, CISM Courses and Lectures, Vol. **302** (Springer-Verlag).

Pagilla, P.R. (2001): *Control of contact problem in constrained Euler-Lagrange cystems*, IEEE Transactions on Automatic Control, vol.**46**, No. 10, 1595-1599.

Pagilla, P.R. & Yu, B. (2004): *An Experimental Study of Planar Impact of a Robot Manipulator*, IEEE/ASME Transactions on Mechatronics, vol.**9**, No. 1, 123-128.

Tornambé, A. (1999): *Modeling and control of impact in mechanical systems: Theory and experimental results*, IEEE Transactions on Automatic Control, vol.**44**, No. 2, 294-309.

van Vliet, J., Sharf, I., & Ma, O. (2000): *Experimental validation of contact dynamics simulation of constrained robotic tasks*, International Journal of Robotics Research, vol.**19**, No.12, 1203-1217.

Wolf, S., & Hirzinger, G. (2008): *A new variable stiffness design: matching requirements of the next robot generation*, Proceedings of the IEEE Int. Conf. on Robotics and Automation, Pasadena, CA, USA, May 19-23, pp.1741-1746.

Xu, W.L., Han, J.D., & Tso, S.K. (2000): *Experimental study of contact transition control incorporating joint acceleration feedback*, IEEE/ASME Transactions on Mechatronics, vol.**5**, No.3, 292-301.

Yu, B. & Pagilla, P.R. (2006): *Design and Implementation of a Robust Switching Control Scheme for a Class of Constrained Robot Tasks*, International Journal of Systems Science, vol.**37**, No. 5, 303-321.