Old and new in stability analysis of Cushing equation: A geometric perspective

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Abstract

This paper focuses on the characterization of the stability crossing curves of Cushing linearized equation. More explicitly, we compute the crossing set, which consists of all frequencies corresponding to all points on the stability crossing curve, and we give their complete classification. Furthermore, the directions in which the zeros cross the imaginary axis are explicitly expressed. A numerical example complete the paper.

1 Introduction

The study considered here is mainly motivated by biological applications including gamma-distributed delays with a gap in population dynamics. In [14], the author largely discusses the connection between gamma distribution delay models, and population dynamics, and a particular attention is paid to the so-called distributed delay with some gap. To the best of the authors' knowledge, the first population dynamics model including gamma-distributed delays is due to Cushing [4], and it received a lot of attention starting with the 80s [3, 1, 2]. The linearized model [3] simply writes as:

$$\dot{x}(t) = -\alpha x(t) + \beta \int_0^t x(t-\theta) g(\theta) d\theta, \tag{1}$$

under appropriate initial conditions. A narrow distribution will lead to some simple discrete delay system of the form $\dot{x}(t) = -\alpha x(t) + \beta x(t-h)$, whose dynamics, and stability are completely understood (see, for instance, [12] and the references therein). Next, if one assumes that the delay kernel is given by the gamma-distribution law:

$$g(\xi) = \frac{a^{n+1}}{n!} \xi^n e^{-a\xi}, \tag{2}$$

the Laplace transform applied to (1), under the definition (2) reduces the stability analysis of (1) to the analysis of some parameter-dependent polynomials of the form:

$$D(s, \bar{\tau}, n) := (s+\alpha)\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} - \beta = 0, \tag{3}$$

where $\bar{\tau} = \dfrac{n+1}{a}$ denotes the corresponding *mean delay* value. One of the problem discussed in [3] was the analysis of the behavior of the roots of the characteristic equation with respect to the imaginary axis when the

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mean delay value $\bar{\tau}$, or the exponent n are varying. The main interest of such a study was to compute the stability regions with respect to the corresponding parameters, and to analyze the sensitivity of such regions when the parameters change. Further discussions on this topics can be found in [14].

Next, Nisbet, and Gurney [18] mention that population dynamics models based on partial differential equations, and reduced for convenience to integro-differential forms are more realistic if the corresponding delay kernel $\hat{g}$ includes some *gap* (see also [1, 14]), that is if it can be expressed as:

$$\hat{g}(\xi) = \begin{cases} 0, & \xi < \tau \\ \frac{a^{n+1}}{n!}(\xi - \tau)^n e^{-a(\xi-\tau)}, & \xi \geq \tau, \end{cases} \tag{4}$$

for some positive delay values τ. Simple computations prove that the corresponding *mean delay* is defined by $\hat{\tau} = \tau + \dfrac{n+1}{a}$. In this case, the stability analysis becomes more complicated, since the parameter-dependent polynomial $D(s, \bar{\tau}, n)$ in (3) becomes a parameter-dependent quasipolynomial of the form (see, for instance, [1, 2]):

$$D(s, \bar{\tau}, \tau, n) := (s + \alpha)\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} - \beta e^{-s\tau} = 0. \tag{5}$$

The paper addresses the problem of analyzing the effects of the gap, and mean delay values on the stability regions of the general characteristic equation (5). Whereas several particular cases received a lot of attention, see, for instance, [3, 1, 2, 14] using various frequency-domain methods, the corresponding methodologies becomes extremely complicated to apply to (5). We think that our approach overcomes this difficulties, and gives a simple, and appealing way to treat such a stability problem. More explicitly, we shall define the *stability crossing curves*, that is the curves consisting of all delays such that the corresponding characteristic quasipolynomial has at least one root on the imaginary axis. Next, we explicitly compute the *crossing set*, that is the set represented by all the frequencies corresponding to all the points in the stability crossing curves, and we discuss the way such a set can be computed as well as its properties. The classification of the stability crossing curves follow naturally from the procedure considered. Finally, we detail the directions in which zeros cross the imaginary axis. To the best of the authors' knowledge, there does not exist any similar analysis in the literature for such a case study.

The main interest of the approach is twofold: first, to understand the underlying mechanisms of stability/instability issues in the case of linear systems including gamma-distributed delays with a gap, and second, to derive some simple stability criteria for such systems. Indeed, it is well known (see, for instance, [5]) that the complete stability characterization of the linear delay systems is still an *open* problem. Furthermore, it was proved in [20], that the problem is NP-hard even in the case of multiple discrete (piece-wise) constant delays. However, the geometry of the stability regions in the delay-parameter space for the two (piece-wise) delays case was completely developed in [8]. The intention of this paper is to give similar insights for this class of dynamical systems with respect to the corresponding gap, and average delay values, respectively. Some illustrative examples complete the approach considered, and give a simple, and easy way to follow the methodology considered.

The remaining paper is organized as follows: Section 2 presents the problem formulation, and some simple prerequisites necessary in developing our results. Next section contains a brief overview over the existing results regarding the stability analysis of Cushing equation. The main results (crossing sets, stability crossing curves classification, tangent and smoothness, crossing directions) and one numerical example are presented in Section 4 and concluding remarks end the paper. The notations are standard.

2 Problem formulation, and preliminaries

As mentioned in the Introduction, our main interest is to analyze the effects of the gap, and mean delay values on the stability regions of the general characteristic equation (5). Consider now the following system, whose dynamics are described by the following characteristic equation:

$$D(s, T, \tau) = (s + \alpha)(1 + sT)^n + \beta \mathrm{e}^{-s\tau} = 0. \tag{6}$$

More explicitly, we study the occurrence of any possible stability switch/reversal [1] resulting by increasing the time delay τ or the average delay T. In other words, we explicitly study the change of number of zeros of (6) on $\mathbb{C}_+$ as the delays (T, τ) vary on $\mathbb{R}_+^2$.
Since the main objective of this study is to identify the regions of (T, τ) in $\mathbb{R}_+^2$ such that $D(s, T, \tau)$ is (asymptotically) stable, we will exclude some cases, and the following assumption appears naturally, as discussed below:

Assumption 1. $\alpha + \beta > 0$*;*

If $\alpha + \beta = 0$ then 0 is a zero of (6) for any $(T, \tau) \in \mathbb{R}_+^2$, and therefore we can never get the stability by increasing T, and τ. For stability at zero delay we require $\alpha + \beta > 0$.

In [11], the authors introduced the notion of *hyperbolicity* for linear delay system. More explicitly, the characteristic equation (6) is said to be *hyperbolic* at some point (T_0, τ_0) if no root of the characteristic equation lies on the imaginary axis for $T = T_0$, and $\tau = \tau_0$.

Using the assumption, and the hyperbolicity notion introduced above, we have the following simple result:

Proposition 1. *The system (6) is hyperbolic for all* $(T, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$ *if and only if:*

$$|\alpha| > |\beta| , \tag{7}$$

Proof: "$\Leftarrow$" It is clear that:

$$| \alpha + j\omega | \geq | \alpha |, \quad \omega \in \mathbb{R}.$$

Next, using (7), it follows:

$$| (1 + j\omega T)^n (\alpha + j\omega) | > | \beta |,$$

for all $(\omega, T) \in \mathbb{R} \times \mathbb{R}_+$. In conclusion, the modulus equation associated to (6) cannot have any solution $j\omega$, with $\omega \in \mathbf{R}^*$, for all $(T, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$, fact which is equivalent to say that the corresponding characteristic equation has no roots on the imaginary axis, excepting eventually the origin.

Let us consider the case at the origin now. Using a simple continuity argument, (7) leads to the inequality $| \alpha | \geq | \beta |$, and thus the only case that one needs to consider is $| \alpha | = | \beta |$, case which is excluded by Assumption 1. In conclusion the hyperbolicity property follows.

"$\Rightarrow$" The argument can be simply done by contradiction, and it is omitted. The proof is completed.

Remark 1. *The proposition above gives a simple* frequency-sweeping *characterization of the so-called* delay-independent hyperbolicity *property. Further discussions on this topics can be found in [17]. In the case, when the system free of delays is asymptotically stable, then the result above gives a very simple condition of* delay-independent *stability (see also [7], and the references therein).*

We can ignore cases where $\alpha < 0$ on biological grounds. So in all that follows we assume $0 < \alpha < \beta$.

3 Stability analysis of Cushing equation: an overview

This section is devoted to existing results in the literature, in the analysis of Cushing model. It is very important to note that even for the simple case without the gap, some of the first results concerning its stability analysis include errors.
In [3] Cooke K.L and Grosmann Z. made a stability analysis using an algebraic approach. They studied the case when $\tau = 0$ so equation (6) becomes $(s + \alpha)(1 + sT)^n + \beta = 0$ and

$$\frac{\mathrm{d}s}{\mathrm{d}T} = -\frac{ns(s + \alpha)}{1 + sT + nT(s + \alpha)}. \tag{8}$$

[1]We are using the same terminology as in Cooke and Grossman [3], that is a root of the characteristic equation *crossing* the imaginary axis, when some parameter is varying.

At the root $s = -j\omega$, if any, we have

$$\frac{\mathrm{d}s}{\mathrm{d}T} = \frac{-nj\omega(j\omega + \alpha)}{nT\alpha + 1 + j(n+1)T\omega} = \frac{n\omega^2 - jn\omega\alpha}{nT\alpha + 1 + j(n+1)T\omega},$$

$$\mathrm{Re}\frac{\mathrm{d}s}{\mathrm{d}T} = \frac{n^2 T\alpha\omega^2 + n\omega^2 - n(n+1)T\alpha\omega^2}{(nT\alpha+1)^2 + (n+1)T^2\omega^2} \Rightarrow$$

$$\mathrm{sign}\left\{\mathrm{Re}\frac{\mathrm{d}s}{\mathrm{d}T}\right\} = \mathrm{sign}[nT\alpha + 1 - (n+1)T\alpha] = \mathrm{sign}(1 - T\alpha).$$

Writing $s = \rho + j\omega$ and using (8) for $T\alpha > 1$ they stated that *any root with a positive real part, if such exists for some T, must cross the imaginary axis and undergo an irreversible change of sign of the real part as T is increased.*

In [1] Blythe and all, corrected the results which characterize the behaviors in presence of a distributed delay. First they stated for linearized Cushing model whose behavior is given by characteristic equation

$$(s+\alpha)\left(1 + s\frac{\tau_1}{n+1}\right)^{n+1} + \beta = 0 \tag{9}$$

that we have a crossing towards instability and there is a range of values of α/β within which restabilisation occurs but beyond which it cannot. Next, they studied a model with a gap given by:

$$(s+\alpha)\left(1 + s\frac{\tau_1}{n+1}\right)^{n+1} + \beta \mathrm{e}^{-s\tau_2} = 0 \tag{10}$$

and they defined the angular quantity θ, by $\tan\theta = \dfrac{\omega\tau_1}{n+1}$.

For

$$P = (\cos y)^{n+1} \cos\left[(n+1)\left(\theta + \frac{\tau_2}{\tau_1}\tan\theta\right)\right] \quad \text{and} \quad M = -(\cos z)^{n+1} \tag{11}$$

where y is a solution of: $\tan\theta + \left(1 + \dfrac{\tau_2}{\tau_1}\sec^2\theta\right)\tan\left[(n+1)\left(\theta + \dfrac{\tau_2}{\tau_1}\tan\theta\right)\right] = 0$ and z is a solution of

$$\theta + \frac{\tau_2}{\tau_1}\tan\theta = \frac{\pi}{n+1}$$

they obtained the restabilisation "window"

$$W(n, \tau_2/\tau_1) = (P - M)/P, \tag{12}$$

which falls to zero as n and/or τ_2/τ_1 increases.

Using another method Boese pointed out in [2] that both papers contain errors and weakness. Boese focussed on the analysis of some models related to

Theorem 1. *The function*

$$f(s) = (s+\alpha)^n + \beta e^{-s\tau} \tag{13}$$

with real a,d,τ with $a > 0$ and $d, \tau \geq 0$ as well as $n \in \mathbb{N}$ is stable if $\tau < \tau(a,d)$ and unstable for $\tau > tau(a,d)$ where

$$\tau(a,d) = \begin{cases} +\infty & d \leq a^n \\ \max\left\{0, \dfrac{\pi - n\arctan(\sqrt{d^{2/n}a^{-2} - 1})}{a\sqrt{d^{2/n}a^{-2} - 1}}\right\} & d > a^n \end{cases} \tag{14}$$

4 Main results

The characterization of the stability crossing curves in the delay parameter space needs the following ingredients: (a) first, the identification of the corresponding crossing points, that is the set of frequencies corresponding to all the points in the stability crossing curves. Next, we define the associated *crossing set*, which will be defined by a finite number of intervals of finite length; (b) second, the classification of the corresponding stability crossing curves, including some simple geometric characterization (tangent, smoothness); (c) finally, the characterization of the way the roots cross the imaginary axis.

All these steps are detailed in the next paragraphs, and one example illustrating our algorithm end this section. The presentation is as simple as possible, and intuitive.

4.1 Identification of crossing points

Let $\mathcal{T}$ denote the set of all $(T,\tau) \in \mathbb{R}_+^2$ such that (6) has at least one zero on imaginary axis. Any $(T,\tau) \in \mathcal{T}$ is known as a *crossing point*. The set $\mathcal{T}$, which is the collection of all crossing points, is known as the *stability crossing curves.*

Based on the results presented in the previous section, it becomes clear that crossing points potentially exist if the condition (7) is not satisfied for some frequency values ω. Such aspects, together with various simple, and intuitive geometrical figures will be addressed in the sequel.

Remark 2. *1) There is $\tau \in \mathbb{R}_+$ which satisfies equation (6) for a fixed $s = j\omega$ if and only if*

$$|(1 + j\omega T)^n|\,|\alpha + j\omega| = |\beta| \tag{15}$$

2) There exists $T \in \mathbb{R}_+$ which satisfies (15) for a fixed $s = j\omega$, $\omega \neq 0$ if and only if

$$|\alpha + j\omega| \leq |\beta| \text{ and } \alpha + j\omega \neq 0 \tag{16}$$

Therefore $\mathcal{T}$ is the set of $(T = T(\omega), \tau = \tau(\omega))$ with ω satisfies (16).

Remark 3. *If ω is a real number and $(T,\tau) \in \mathbb{R}_+^2$ then*

$$(-j\omega + \alpha)(1 - j\omega T)^n + \beta e^{j\omega\tau} = \overline{(\alpha + j\omega)(1 + j\omega T)^n + \beta e^{-j\omega\tau}}$$

Therefore we only need to consider positive ω. Let Ω be the set of all positive real number which satisfy (16). We will refer to Ω as the crossing set. *It contains all the ω such that some zero(s) of $D(s,T,\tau)$ may cross the imaginary axis at $j\omega$.*

Remark 4. *If $D(j\omega,T,\tau) = 0$ then $D(j\omega,T,\tau + 2k\pi) = 0$, $\forall k \in \mathbb{Z}$. In this context exists $\tau_0 \in (-\pi,\pi)$ such that $D(j\omega,T,\tau_0) = 0$.*

Proposition 2. *The following statements are true:*

1) The crossing set Ω consists of one interval $\left(0, \sqrt{\beta^2 - \alpha^2}\right]$ of finite length.

2) $\displaystyle\lim_{\omega \to \sqrt{\beta^2-\alpha^2}} T = 0$

3) $\displaystyle\lim_{\omega \to 0} T$ *and* $\displaystyle\lim_{\omega \to 0} \tau_k$ *are infinite.*

Proof. 1) From definition Ω is the set of all ω which satisfy (16). Using Remark 3, Ω is the set of all positive ω which satisfy $\omega \leq \sqrt{\beta^2 - \alpha^2}$. Therefore $\Omega = \left(0, \sqrt{\beta^2 - \alpha^2}\right]$.

For each given $\omega_* \in \Omega$ we may easily find all the corresponding pairs (T,τ) satisfying (6) as follows:

$$T = \frac{1}{\omega_*}\left(\left|\frac{\beta}{\alpha + j\omega_*}\right|^{\frac{2}{n}} - 1\right)^{\frac{1}{2}}, \quad \tau_k = \frac{1}{\omega_*}\left(\arg\frac{-\beta}{(1 + j\omega_* T)^n(\alpha + j\omega_*)} + 2k\pi\right),$$

where $k \in \mathbb{Z}$ such that $\tau_k > 0$.

2) Obvious $\lim\limits_{\omega \to \sqrt{\beta^2-\alpha^2}} T = \lim\limits_{\omega \to \sqrt{\beta^2-\alpha^2}} \frac{1}{\omega}\left(\left|\frac{\beta}{\alpha + j\omega}\right|^{\frac{2}{n}} - 1\right)^{\frac{1}{2}} = 0.$

3)

$$\lim_{\omega \to 0} T = \lim_{\omega \to 0} \frac{1}{\omega}\left(\left|\frac{\beta}{\alpha + j\omega}\right|^{\frac{2}{n}} - 1\right)^{\frac{1}{2}} = \infty$$

and

$$\lim_{\omega \to 0} \tau_k = \lim_{\omega \to 0} \frac{1}{\omega}\left(\arg \frac{-\beta}{(1 + j\omega T)^n(\alpha + j\omega)} + 2k\pi\right) = -\frac{1}{\omega}\arctan\frac{\omega}{\alpha} - \frac{n}{\omega}\arctan \omega T + \frac{2k\pi}{\omega} = \infty$$

Proposition 3. *The following monotonicity properties are true:*

1) $T = T(\omega)$ is a decreasing function on Ω;

2) $\tau = \tau(\omega)$ is a decreasing function on Ω;

3) $\tau = \tau(T)$ is a increasing function on $(0, \infty)$.

Proof. 1) Let $0 < \omega_1 < \omega_2 < \sqrt{\beta^2 - \alpha^2} \Rightarrow \begin{cases} \omega_1^2 + \alpha^2 < \omega_2^2 + \alpha^2 \Rightarrow \dfrac{\beta^2}{\omega_1^2 + \alpha^2} > \dfrac{\beta^2}{\omega_2^2 + \alpha^2} \\ \dfrac{1}{\omega_1} > \dfrac{1}{\omega_2} \end{cases} \Rightarrow$

$$T(\omega_1) = \frac{1}{\omega_1}\left(\left|\frac{\beta}{\alpha + j\omega_1}\right|^{\frac{2}{n}} - 1\right)^{\frac{1}{2}} > \frac{1}{\omega_2}\left(\left|\frac{\beta}{\alpha + j\omega_2}\right|^{\frac{2}{n}} - 1\right)^{\frac{1}{2}} = T(\omega_2)$$

2) Using the formula of τ_k given in previous proposition we can write

$$\tau_k = \frac{1}{\omega}\left[(2k+1)\pi - \arg \beta - \arctan\frac{\omega}{\alpha} - n \arctan \omega T\right]$$

and it is easy to show that τ_k is decreasing.
3) $T = T(\omega) : \Omega \mapsto (0, \infty)$ is decreasing continuous function and $\lim\limits_{\omega \to 0} T = \infty$, $\lim\limits_{\omega \to \sqrt{\beta^2-\alpha^2}} T = 0$ so $\omega = \omega(T) : (0, \infty) \mapsto \Omega$ is decreasing continuous function. Therefore $\tau = \tau(T)$ is increasing function on $(0, \infty)$.

Example 1. *Consider a system with*

$$P(s) = s + 3 \text{ and } Q(s) = 5 \tag{17}$$

Figure 1 plots $\frac{|P(j\omega)|}{|Q(j\omega)|}$ against ω. The crossing set Ω can be easily identified from the Figure 1, it contains one interval

$\Omega = (0, 4]$ *of type 02,1*

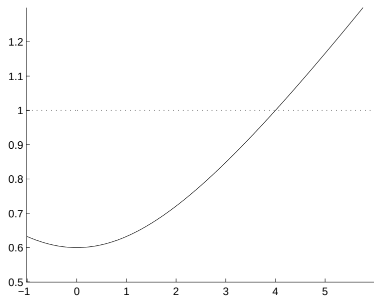

Figure 1: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ versus ω for the system represented by (17)

4.2 Tangents and smoothness

Next, we will discuss the smoothness of the curves in $\mathcal{T}$. In this part we use an approach based on implicit function theorem. For this purpose we consider T and τ as implicit functions of $s = j\omega$ defined by (6). As s moves to imaginary axis, $(T,\tau) = (T(\omega), \tau(\omega)$ moves along the $\mathcal{T}$. For a given $\omega \in \Omega$, let

$$R_0 = Re\left(\frac{j}{s}\frac{\partial D(s,T,\tau)}{\partial s}\right)_{s=j\omega} =$$

$$\frac{1}{\omega}Re\left\{\left[nT(\alpha+j\omega)+(1+j\omega T)\right](1+j\omega T)^{n-1} - \tau\beta \mathrm{e}^{-j\omega\tau}\right\}$$

$$I_0 = Im\left(\frac{j}{s}\frac{\partial D(s,T,\tau)}{\partial s}\right)_{s=j\omega} =$$

$$\frac{1}{\omega}Im\left\{\left[nT(\alpha+j\omega)+(1+j\omega T)\right](1+j\omega T)^{n-1} - \tau\beta \mathrm{e}^{-j\omega\tau}\right\}$$

and

$$R_1 = Re\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial T}\right)_{s=j\omega} = Re\left(n(1+j\omega T)^{n-1}(\alpha+j\omega)\right)$$

$$I_1 = Im\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial T}\right)_{s=j\omega} = Im\left(n(1+j\omega T)^{n-1}(\alpha+j\omega)\right)$$

$$R_2 = Re\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial \tau}\right)_{s=j\omega} = -Re\left(\beta \mathrm{e}^{-j\omega\tau}\right)$$

$$I_2 = Im\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial \tau}\right)_{s=j\omega} = -Im\left(\beta \mathrm{e}^{-j\omega\tau}\right)$$

Then, since $D(s,T,\tau)$ is an analytic function of s, T and τ, the implicit function theorem indicates that the tangent of $\mathcal{T}$ can be expressed as

$$\begin{pmatrix} \frac{\mathrm{d}T}{\mathrm{d}\omega} \\ \frac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - I_0 R_2 \\ I_0 R_1 - R_0 I_1 \end{pmatrix}, \tag{18}$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{19}$$

It follows that $\mathcal{T}$ is smooth everywhere except possibly at the points where either (19) is not satisfied, or when

$$\frac{\mathrm{d}T}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{20}$$

Proposition 4. *The curves in $\mathcal{T}$ are smooth everywhere except possibly at the degenerate points corresponding to ω in any one of the following cases:*

1) $s = j\omega$ is a multiple solution of (6)

2) $\omega = \sqrt{\beta^2 - \alpha^2}$.

Proof. If (20) is satisfied then $s = j\omega$ is a multiple solution of (6).

Condition (19) is not satisfied if and only if $\frac{I_1}{R_1} = \frac{I_2}{R_2} \Leftrightarrow$
$\arg\left(n(1+j\omega T)^{n-1}(\alpha + j\omega)\right) = \arg\left(-\beta \mathrm{e}^{-j\omega\tau}\right) \Leftrightarrow \frac{1+j\omega T}{|1+j\omega T|} = 1 \Leftrightarrow$
$T = 0 \Leftrightarrow (\alpha + j\omega) + \beta \mathrm{e}^{-j\omega\tau} = 0 \Leftrightarrow |\alpha + j\omega| = |\beta|.$

4.3 Direction of crossing

Next we will discuss the direction in which the solutions of (6) cross the imaginary axis as (T, τ) deviates from a curve in $\mathcal{T}$. We will call the direction of the curve that corresponds to increasing ω the *positive direction.* Notice, as the curve passes through the points corresponding to the end points of Ω,the positive direction is reversed. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left.* Again, due to the possible reversion of parametrization the same region may be considered on the left with respect to one point of the curve,and on the right with respect to another point of the curve.

To establish the direction of crossing we need to consider T and τ as functions of $s = \sigma + j\omega$ i.e.,function of two real variables σ and ω, and partial notation needs to be adopted instead. Since the tangent of $\mathcal{T}$ along the positive direction is $\left(\frac{\partial T}{\partial \omega}, \frac{\partial \tau}{\partial \omega}\right)$, the normal to $\mathcal{T}$ pointing to the left hand side of positive direction is $\left(-\frac{\partial \tau}{\partial \omega}, \frac{\partial T}{\partial \omega}\right)$. The crossing of a pair of conjugate complex solutions of (6) is given by the moving of (T, τ) along the direction $\left(\frac{\partial T}{\partial \sigma}, \frac{\partial \tau}{\partial \sigma}\right)$. So, if a pair of conjugate complex solutions of (6) cross imaginary axis to the right half plane then:

$$\left(\frac{\partial T}{\partial \omega}\frac{\partial \tau}{\partial \sigma} - \frac{\partial \tau}{\partial \omega}\frac{\partial T}{\partial \sigma}\right)_{s=j\omega} > 0 \tag{21}$$

i.e. the region on the left of $\mathcal{T}$ at ω has two more solutions in right half plane. If the inequality (21) is reversed then the region on the left of $\mathcal{T}$ at ω has two fewer solutions in right half plane. Like in (18) we can express

$$\begin{pmatrix} \frac{\mathrm{d}T}{\mathrm{d}\sigma} \\ \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} \end{pmatrix}_{s=j\omega} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} I_0 \\ -R_0 \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 R_2 + I_0 I_2 \\ -R_0 R_1 - I_0 I_1 \end{pmatrix}, \tag{22}$$

where R_i and I_i are defined in previous section. Using this we can arrive to the following result:

Proposition 5. *Let $\omega \in \Omega$ and $(T, \tau) \in \mathcal{T}$. Then a pair of solutions of (6) cross the imaginary axis to the right, through the "gates" $\pm j\omega$ if $R_2 I_1 - R_1 I_2 > 0$, and cross to the left if the inequality is reversed.*

Proof. Easy computation shows that

$$\left(\frac{\partial T}{\partial \omega}\frac{\partial \tau}{\partial \sigma} - \frac{\partial \tau}{\partial \omega}\frac{\partial T}{\partial \sigma}\right)_{s=j\omega} = \frac{(R_0^2 + I_0^2)(R_2 I_1 - R_1 I_2)}{(R_1 I_2 - R_2 I_1)^2}$$

Therefore (21) can be written as $R_2 I_1 - R_1 I_2 > 0$.

Example 2 (linearized Cushing equation with a gap)**.** *In this example we apply the above method for the Cushing linearized equation* $(s+a)(1+sT)^n + be^{-s\tau} = 0$*. First it's easy to remark that the only interesting case is* $|a| < |b|$*. All other cases don't present any stability switch because the crossing set* Ω *is empty.*
If $|a| < |b|$ *then* $\Omega = (0, \sqrt{b^2 - a^2}\ ]$ *and the corresponding pairs* (T, τ) *are given by:*

$$T = \frac{1}{\omega}\left[\left(\frac{b^2}{\omega^2 + a^2}\right)^{1/n} - 1\right]^{1/2}, \quad \tau_i = \frac{1}{\omega}\left[\arg\left(\frac{-b}{(a+j\omega)(1+j\omega T)^n}\right) + 2i\pi\right]$$

According with the Proposition 2 we get $\lim_{\omega \to \sqrt{b^2-a^2}} T = 0$, $\lim_{\omega \to \sqrt{b^2-a^2}} \tau_i = \frac{1}{\sqrt{b^2 - a^2}}\left(2i\pi + \arg\frac{-b}{a} - \arctan\frac{\sqrt{b^2 - a^2}}{a}\right)$, $\lim_{\omega \to 0} T = \infty$ *and* $\lim_{\omega \to 0} \tau_i = \infty$*. Also the slopes of the corresponding asymptotes are given by*

$$\lim_{\omega \to 0}\frac{\tau}{T} = \frac{-n\arctan\left[\left(\frac{b^2}{a^2}\right)^{1/n} - 1\right]^{1/2} + \arg\frac{-b}{a} + 2i\pi}{\left[\left(\frac{b^2}{a^2}\right)^{1/n} - 1\right]^{1/2}}$$

The following picture plots τ_k, $k \in \{0,1,2,3,4\}$ *against* T *in the case* $n = 1$ *and* $n = 4$ *for* $a = 3$ *and* $b = 5$*. We can easily see that* $\tau_{k+1}(\omega) > \tau_k(\omega)$, $\forall k > i_0$ *and* $\omega \in \Omega$*.*

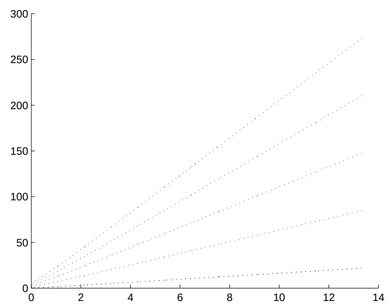

Figure 2: τ_k, $k \in \{0,1,2,3,4\}$ versus T when $n = 1$

Proposition 6. *For the previous system all the crossing directions of the characteristic roots are towards instability.*

Proof *We can easily compute*

$$\left.\frac{ds}{d\tau}\right|_{s=j\omega} = \frac{j\omega b e^{-j\omega\tau}}{(1+j\omega T)^n + nT(j\omega + a)(1+j\omega T)^{n-1} - b\tau e^{-j\omega\tau}} \tag{23}$$

and then

$$sgn\ Re\left.\left(\frac{ds}{d\tau}\right)^{-1}\right|_{s=j\omega} = sgn\left(\frac{\omega}{a^2+\omega^2} + \frac{n\omega T^2}{1+\omega^2 T^2}\right) > 0 \tag{24}$$

and the proof is complete.

So, after the first cross the stability is lost and never regained. Therefore, for all n*, we have only one stability region delimited by* $\mathcal{T}^0$*.*

5 Concluding remarks

In this paper we have characterized the geometry of the stability crossing curves in the parameter space. Our approach is easier than other existing approach. The presentation is as simple as possible, and intuitive. We intend to adapt this method to more general cases and give various applications.

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