# STABILITY CROSSING CURVES OF SHIFTED GAMMA-DISTRIBUTED DELAY SYSTEMS

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**Abstract.** This paper characterizes the stability crossing curves of a class of linear systems with gamma-distributed delay with a gap. First, we describe the crossing set, i.e., the set of frequencies where the characteristic roots may cross the imaginary axis as the parameters change. Then, we describe the corresponding stability crossing curves, i.e., the set of parameters such that there is at least one pair of characteristic roots on the imaginary axis. Such stability crossing curves divide the parameter space $\mathbb{R}_+^2$ defined by the mean delay and the gap into different regions. Within each such region, the number of characteristic roots on the right half complex plane is fixed. This naturally describes the regions of parameters where the system is stable. The classification of the stability crossing curves is also discussed. Some illustrative examples (Cushing equation in biology, traffic flow models in transportation systems, control over networks of a simplified helicopter model) are also presented.

**Key words.** stability; crossing curves; distributed delay; quasipolynomial; gamma-distribution.

**AMS subject classifications.** 34K20, 34D99, 93D09, 93D99.

**1. Introduction.** The stability of dynamical systems in the presence of time delay is a problem of recurring interest (see, for instance, [11], [16], [8], [14], and the references therein). The presence of a time delay may induce instabilities and complex behaviors. Systems with distributed delays are present in many scientific disciplines such as physiology, population dynamics, and engineering.

One of the first studies devoted to population dynamics using a model with gamma-distributed delay is due to Cushing [5]. The linearization of this model is

$$\dot{x}(t) = -\alpha x(t) + \beta \int_{-\infty}^{t} g(t-\theta)x(\theta)d\theta, \qquad (1.1)$$

where $\alpha$ is a constant defining the death rate per unit time, and $\beta$ a constant corresponding to the maternity function. The integration kernel of the distributed delay is the gamma distribution [15, 4]

$$g(\xi) = \frac{a^{n+1}}{n!}\xi^n e^{-a\xi}. \qquad (1.2)$$

Applying a Laplace transform of (1.1), with $g(\xi)$ as expressed in (1.2) yields a parameter-dependent polynomial characteristic equation of the form

$$D(s;\ \bar{\tau}, n) := (s+\alpha)\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} - \beta = 0, \qquad (1.3)$$

where "$s$" is the Laplace transform variable and $\bar{\tau} = (n+1)/a$ is the *mean delay*.

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Cooke and Grossman [4] discussed the change of stability of (1.3) when one of the parameters, the mean delay value $\bar{\tau}$ or the exponent $n$, varies while the other is fixed.

Nisbet and Gurney [17] modified the gamma distribution $g(\xi)$ expressed in (1.2) to the gamma distribution with a gap

$$\hat{g}(\xi) = \begin{cases} 0, & \xi < \tau \\ \frac{a^{n+1}}{n!}(\xi - \tau)^n e^{-a(\xi - \tau)}, & \xi \geq \tau, \end{cases} \tag{1.4}$$

to more accurately reflect the reality (see, for instance, [1], [15] for additional discussions). In this case, a simple computation shows that the mean delay is $\hat{\tau} = \tau + \dfrac{n+1}{a}$. The characteristic equation becomes a parameter-dependent quasipolynomial equation [1], [2] of the form

$$\hat{D}(s;\ \bar{\tau}, \tau, n) := (s + \alpha)\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} - \beta e^{-s\tau} = 0. \tag{1.5}$$

We note that [2] pointed out some inaccuracies of the earlier results presented in [4] and [1]. It is also interesting to mention that Farkas, et. al [7] studied the bifurcation problem of predator-prey model, also in the case of a gamma-distributed delay. More general bifurcation study of systems with distributed delay can be found in a book by Farkas [6].

More recently and in a quite different field (engineering), it was pointed out that such gamma-distributed delays with a gap can also be encountered in the problem of controlling objects over communication networks [19]. More specifically, the overall communication delay in the network is modeled by a gamma-distributed delay with a gap, where the gap value corresponds to the minimal propagation delay in the network, which is always strictly positive. The stability problem of the closed-loop system in [19] reduces to a parameter-dependent characteristic quasipolynomial equation of the following form,

$$D(s;\ \bar{\tau}, \tau, n) := P(s)\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} + Q(s)e^{-s\tau} = 0, \tag{1.6}$$

where $P(s)$ and $Q(s)$ are polynomials. Obviously, the equation (1.5) is a special case of (1.6).

Another research area where a distributed delay appears naturally is the traffic flow dynamics. A simplified car-following model, where multiple vehicles in a ring have drivers with identical behavior, and under the influence of a single constant time-delay [3, 12], can be written as

$$\dot{x}_i(t) = \alpha_i(x_{i-1}(t - \tau) - x_i(t - \tau)),\ i = 1, \ldots, p, \tag{1.7}$$

where $p$ is the number of vehicles considered and $x_0 = x_p$. The left hand side represents the acceleration of the $i^{th}$ vehicle, and the right hand side express the velocity difference of consecutive vehicles.

When the delays are not assumed to be identical, several models in the literature are used to describe the dynamics of the model (see, for instance, [21] for some classifications and a large list of references therein). As suggested in [22], we can extend the previous models by incorporating a more general memory effect. Consider the following system:

$$\dot{x}_i(t) = \alpha_i \int_{-\infty}^{t} g(t - \theta)(x_{i-1}(\theta) - x_i(\theta))\mathrm{d}\theta, \tag{1.8}$$

where $g$ is the delay distribution, which can represent both dead-time and past memory. The corresponding characteristic equation of (1.8) is given by

$$\det[sI - (A_1 + A_2)G(s)] = 0, \tag{1.9}$$

where $G$ denotes the Laplace transform of $g$. When $g$ represents the gamma-distribution with a gap, we get

$$G(s) = e^{-s\tau} \left(1 + s\frac{\bar{\tau}}{n+1}\right)^{-(n+1)}.$$

In the simplest case of two vehicles in a ring ($p = 2$, $i = 1, 2$, and $x_0 = x_2$), the matrices $A_1$ and $A_2$ are given by

$$A_1 = \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}, \tag{1.10}$$

and the characteristic equation becomes

$$s\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} + (\alpha_1 + \alpha_2)e^{-s\tau} = 0, \tag{1.11}$$

which is again a special case of equation (1.6).

Finally, another interesting engineering example involving gamma-distributed delay is the machine tool vibration problem. The readers are referred to the nice paper by Stépán [24] for details. It is also interesting to mention that Insperger and Stépán [13] also used gamma-distributed delay in their numerical study of time-delay system.

In this paper, we will study the stability of systems with the characteristic equation (1.6) as the parameters $\bar{\tau}$ and $\tau$ vary. Specifically, we will describe the stability crossing curves, i.e., the set of parameters such that there exists at least one pair of characteristic roots on the imaginary axis. Such stability crossing curves divide the parameter space $\mathbb{R}_+^2$ into different regions. Within each such region, the number of characteristic roots on the right-half complex plane is fixed. This naturally describes the regions of parameters where the system is stable.

It should be noted that there have been numerous works in the literature to describe the stability regions of parameter space, known as stability charts [23], [24]. These descriptions are typically valid for one specific system except that the parameters are allowed to vary. In a recent paper, Gu, et al [9] gave a characterization of the stability crossing curves for systems with two discrete delays as the parameters. One significant difference of [9] as compared to the stability charts is the fact that such characterization applies to any systems within the class, i.e., any system with two delays. We note also the paper of [20], where we can find some interesting characterization that can be used for a large class of time delay systems (including distributed delay systems). However, the approach proposed in [20] requires rather heavy computation when dealing with quasipolynomials of a high degree.

The current paper follows the line of [9], and our conclusion is valid for any system of the form (1.6).

The rest of this paper is organized as follows: Section 2 contains the problem statement and assumptions. Section 3 first discusses geometric characterization of the crossing curves. Next, the stability crossing curves together with their classification are described. Several simple examples are described to illustrate the types of

curves in our classification. Finally, tangent and smoothness properties and crossing direction are described. Section 4 includes four illustrative examples in some detail: linearized first-order Cushing equation, a second-order system depicting some particular behavior, a simplified traffic flow model, and a control over networks of a simplified helicopter model. Some concluding remarks end the paper.

Throughout the paper, the following standard notation is used: $\mathbb{C}$ ($\mathbb{C}^+$, $\mathbb{C}^-$) is the set of complex numbers (with strictly positive, and strictly negative real parts), and $j = \sqrt{-1}$. For $z \in \mathbb{C}$, $\angle(z)$, $Re(z)$ and $Im(z)$ define the argument, the real part and the imaginary part of $z$. $\mathbb{R}$ ($\mathbb{R}^+$, $\mathbb{R}^-$) denotes the set of real numbers (greater than or equal to zero, less than or equal to zero). Next, $\mathbb{N}$ is the set of natural numbers, including zero and $\mathbb{Z}$ the set of integers. Finally, RHP denotes the right-half plane of $\mathbb{C}$.

**2. Problem formulation.** Consider a system with the following characteristic equation

$$D(s;\ T,\tau) := P(s)(1 + sT)^n + Q(s)\mathrm{e}^{-s\tau} = 0, \qquad (2.1)$$

where the two parameters $T$ and $\tau$ are nonnegative. We will describe the *stability crossing curves* $\mathcal{T}$, which is the set of $(T,\tau)$ such that (2.1) has imaginary solutions.

As the parameters $(T,\tau)$ cross the stability crossing curves, some characteristic roots cross the imaginary axis. Therefore, the number of roots on the right half complex plane are different on the two sides of the curves, from which, we may describe the parameter regions of $(T,\tau)$ in $\mathbb{R}^2_+$ for the system to be stable.

Another related useful concept is the *crossing set* $\Omega$, which is defined as the collection of all $\omega > 0$ such that there exists a parameter pair $(T,\tau)$ such that $D(j\omega;\ T,\tau) = 0$. In other words, as the parameters $T$ and $\tau$ vary, the characteristic roots may cross the imaginary axis at $j\omega$ if and only if $\omega \in \Omega$.

We will restrict our discussions on the systems that satisfy the following assumptions.

**Assumption I.** $\deg(Q) < \deg(P)$;
**Assumption II.** $P(0) + Q(0) \neq 0$;
**Assumption III.** $P(s)$ and $Q(s)$ do not have common zeros;
**Assumption IV.** If $P(s) = p$, $Q(s) = q$, where $p$ and $q$ are constant real, then $|p| \neq |q|$;
**Assumption V.** $P(0) \neq 0$, $|P(0)| \neq |Q(0)|$;
**Assumption VI.** $P'(j\omega) \neq 0$ whenever $P(j\omega) = 0$.

Assumption I means that the time-delay system represented by (2.1) is of retarded type. While not discussed here, it is possible to extend the analysis to systems of neutral type by relaxing this assumption to also allow $\deg(Q) = \deg(P)$, as long as $\lim_{s\to\infty} Q(s)/P(s) < 1$ is satisfied. Assumption II is made to exclude some trivial cases. If it is not satisfied, then $s = 0$ is a solution of (2.1) for arbitrary $(T,\tau)$, and therefore, the system can never be stable. Regarding Assumption III, if it is violated, we may find a common factor of the highest order $c(s) \neq$ constant of $P(s)$ and $Q(s)$. This would indicates that $D(s;\ T,\tau) = c(s)\hat{D}(s;\ T,\tau)$, where $\hat{D}(s;\ T,\tau)$ satisfies Assumption III, and our analysis can still proceed on $\hat{D}(s;\ T,\tau)$. Finally, Assumptions IV to VI are made to exclude some rare singular cases in order to simplify presentation.

Notice, we have restricted any element $\omega$ of the crossing set $\Omega$ to satisfy $\omega > 0$. Indeed, the discussion of $\omega < 0$ is redundant in view of the fact that $D(-j\omega;\ T,\tau)$ is

the complex conjugate of $D(j\omega; T, \tau)$. Also, $\omega = 0$ is never an element of $\Omega$ in view of Assumption II.

## 3. Main results.

**3.1. Crossing set and stability crossing curves.** Consider a fixed $\omega > 0$, we first observe that as $T$ and $\tau$ each vary within $[0, \infty)$, i.e., $(T, \tau)$ vary in $\mathbb{R}_+^2$, $|1 + j\omega T|^n \in [1, \infty)$, $|e^{j\omega\tau}| = 1$, and $\angle e^{j\omega\tau}$ may assume any nonnegative value by choosing appropriate $\tau$. From this observation, it is not difficult to conclude the following proposition.

PROPOSITION 3.1. *Given any $\omega > 0$, $\omega \in \Omega$ if and only if it satisfies*

$$0 < |P(j\omega)| \le |Q(j\omega)|, \tag{3.1}$$

*and all the corresponding $T, \tau$ can be calculated by*

$$T = \frac{1}{\omega}\left(\left|\frac{Q(j\omega)}{P(j\omega)}\right|^{2/n} - 1\right)^{1/2}, \tag{3.2}$$

$$\tau = \tau_m = \frac{1}{\omega}(\angle Q(j\omega) - \angle P(j\omega) - n\arctan(\omega T) + \pi + m2\pi), \tag{3.3}$$
$$m = 0, \pm 1, \pm 2, ....$$

**Proof.** For necessity of (3.1), let $\omega \in \Omega$, and apply modulus to (2.1), we obtain

$$|(1 + j\omega T)^n|\,|P(j\omega)| = |Q(j\omega)|. \tag{3.4}$$

This implies $|P(j\omega)| \le |Q(j\omega)|$, because $|(1 + j\omega T)^n| \ge 1$. In addition, $|P(j\omega)| > 0$ is also necessary. Otherwise, $P(j\omega) = 0$, which implies $Q(j\omega) = 0$ in view of (3.4). But this violates Assumption III.

For sufficiency of (3.1), we only need to recognize that $T$ and $\tau$ given by (3.2) and (3.3) make $s = j\omega$ a solution of (2.1). It is also easy to see by direct solution that $T$ and $\tau$ given by (3.2) and (3.3) are all the solutions. $\square$

There are only a finite number of solutions to each of the following two equations

$$P(j\omega) = 0, \tag{3.5}$$

and

$$|P(j\omega)| = |Q(j\omega)|, \tag{3.6}$$

because $P$ and $Q$ are both polynomials satisfying Assumptions I to IV. Therefore, $\Omega$, which is the collection of $\omega$ satisfying (3.1), consists of a finite number of intervals. Denote these intervals as $\Omega_1, \Omega_2, ..., \Omega_N$. Then

$$\Omega = \bigcup_{k=1}^{N} \Omega_k.$$

Without loss of generality, we may order these intervals from left to right, i.e., for any $\omega_1 \in \Omega_{k_1}$, $\omega_2 \in \Omega_{k_2}$, $k_1 < k_2$, we have $\omega_1 < \omega_2$.

In order to give a geometric interpretation that allows deriving the crossing set $\Omega$, for $s = j\omega$, we rewrite (2.1) as

$$\left(-\frac{Q(j\omega)}{P(j\omega)}\right)^{1/n} e^{-j\omega\tau/n} = 1 + j\omega T \qquad (3.7)$$

The equation (3.7) can be interpreted as the intersection between a circle with radius $|Q(j\omega)/P(j\omega)|^{1/n}$ and a vertical line passing trough the point $(1,0)$, in the complex plane. Therefore, the characterization of $\Omega$ can be easily derived from the following figure: We will not restrict $\angle Q(j\omega)$ and $\angle P(j\omega)$ to a $2\pi$ range. Rather, we allow
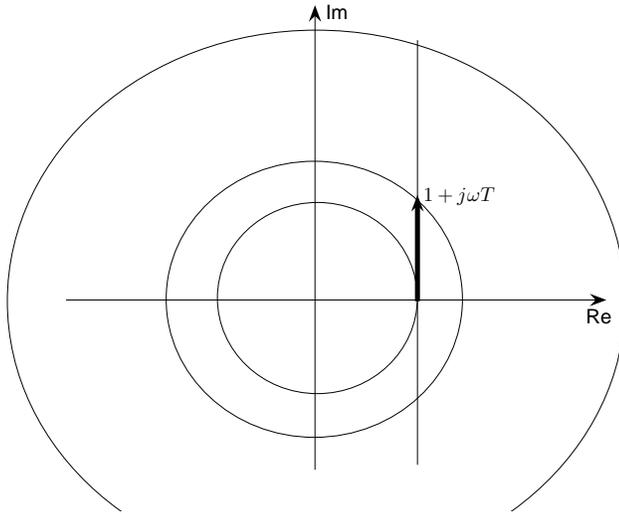


FIG. 3.1. *The intersection is possible only if the radius $|Q(j\omega)/P(j\omega)|$, is larger than 1. The extreme cases for intersection are given by $|Q(j\omega)/P(j\omega)| = 1$ or $|Q(j\omega)/P(j\omega)| \to \infty \Leftrightarrow P(j\omega) \to 0$*

them to vary continuously within each interval $\Omega_k$. Thus, for each fixed $m$, (3.2) and (3.3) represent a continuous curve. We denote such a curve as $\mathcal{T}_k^m$. Therefore, corresponding to a given interval $\Omega_k$, we have an infinite number of continuous stability crossing curves $\mathcal{T}_k^m$, $m = 0, \pm1, \pm2, ....$. It should be noted that, for some $m$, part or the entire curve may be outside of the range $\mathbb{R}_+^2$, and therefore, may not be physically meaningful.

The collection of all the points in $\mathcal{T}$ corresponding to $\Omega_k$ may be expressed as

$$\mathcal{T}_k = \bigcup_{m=-\infty}^{+\infty} \left(\mathcal{T}_k^m \bigcap \mathbb{R}_+^2\right).$$

Obviously, $\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}_k$.

**3.2. Classification of stability crossing curves.** Let the left and right end points of interval $\Omega_k$ be denoted as $\omega_k^\ell$ and $\omega_k^r$, respectively. Due to Assumptions IV and V, it is not difficult to see that each end point $\omega_k^\ell$ or $\omega_k^r$ must belong to one, and only one, of the following three types:

**Type 1.** It satisfies the equation (3.6).

**Type 2.** It satisfies the equation (3.5).

**Type 3.** It equals 0.

Denote an end point as $\omega_0$, which may be either a left end or a right end of an interval $\Omega_k$. Then the corresponding points in $\mathcal{T}_k^m$ may be described as follows.

If $\omega_0$ is of type 1, then $T = 0$. In other words, $\mathcal{T}_k^m$ intersects the $\tau$-axis at $\omega = \omega_0$.

If $\omega_0$ is of type 2, then as $\omega \to \omega_0$, $T \to \infty$ and

$$\tau \to \frac{1}{\omega_0}\left( \angle Q(j\omega_0) - \lim_{\omega \to \omega_0} \angle P(j\omega) - \frac{n\pi}{2} + \pi + m2\pi \right). \tag{3.8}$$

Obviously,

$$\lim_{\omega \to \omega_0} \angle P(j\omega) = \angle \left[ \frac{d}{d\omega} P(j\omega) \right]_{\omega \to \omega_0} \tag{3.9}$$

if $\omega_0$ is the left end point $\omega_k^\ell$ of $\Omega_k$, and

$$\lim_{\omega \to \omega_0} \angle P(j\omega) = \angle \left[ \frac{d}{d\omega} P(j\omega) \right]_{\omega \to \omega_0} + \pi \tag{3.10}$$

if $\omega_0$ is the right end point $\omega_k^r$ of $\Omega_k$. In other words, $\mathcal{T}_k^m$ approaches a horizontal line.

Obviously, only $\omega_1^\ell$ may be of type 3. Due to non-singularity assumptions, if $\omega_1^\ell = 0$, we must have $0 < |P(0)| < |Q(0)|$. In this case, as $\omega \to 0$, both $T$ and $\tau$ approach $\infty$. In fact, $(T, \tau)$ approaches a straight line with slope

$$\tau/T \to \frac{(\angle Q(0) - \angle P(0) - n \arctan \alpha + \pi + m2\pi)}{\alpha}, \tag{3.11}$$

where

$$\alpha = \left( \left| \frac{Q(0)}{P(0)} \right|^{2/n} - 1 \right)^{1/2}.$$

We say an interval $\Omega_k$ is of type $\ell r$ if its left end is of type $\ell$ and its right end is of type $r$. We may accordingly divide these intervals into the following 6 types.

**Type 11.** In this case, $\mathcal{T}_k^m$ starts at a point on the $\tau$-axis, and ends at another point on the $\tau$-axis.

**Type 12**. In this case, $\mathcal{T}_k^m$ starts at a point on the $\tau$-axis, and the other end approaches $\infty$ along a horizontal line.

**Type 21**. This is the reverse of type 12. $\mathcal{T}_k^m$ starts at $\infty$ along a horizontal line, and ends at the $\tau$-axis.

**Type 22**. In this case, both ends of $\mathcal{T}_k^m$ approaches horizontal lines.

**Type 31**. In this case, $\mathcal{T}_k^m$ begins at $\infty$ with an asymptote of slope expressed in (3.11). The other end is at the $\tau$-axis.

**Type 32**. In this case, $\mathcal{T}_k^m$ again begins at $\infty$ with an asymptote of slope expressed in (3.11). The other end approaches $\infty$ along a horizontal line.

In the sequel, we present two *academic* examples to illustrate some cases discussed above.

EXAMPLE 3.1. (**Type 11**) *Let $n = 1$, $P(s) = s^2 + 3s + 2$ and $Q(s) = \sqrt{10}s$. Figure 3.2 (left) plots $|P(j\omega)|/|Q(j\omega)|$ against $\omega$. From the plot, it can be seen that the*
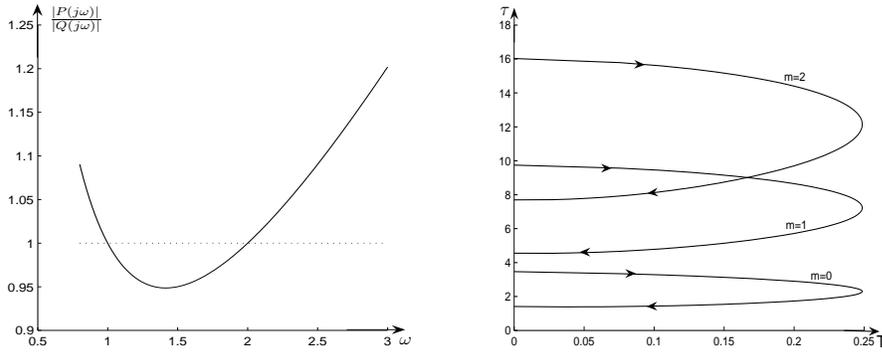
FIG. 3.2. *Example 3.1. Left: $|P(j\omega)|/|Q(j\omega)|$ against $\omega$. Right: $\mathcal{T}_1^m$, for $m = 0, 1, 2$ (type 11) .*

crossing set $\Omega$ contains only one interval $\Omega = \Omega_1 = [1, 2]$ of type 11. Correspondingly, the stability crossing curves $\mathcal{T}$ is shown in Figure 3.2 (right), which consists of a series of curves with both ends on the $\tau$-axis.

EXAMPLE 3.2. (**Type 22** and **32**) *Figure 3.3 (left) plots $|P(j\omega)|/|Q(j\omega)|$ against $\omega$ with $n = 1$,*

$$P(s) = s^4 + 3s^2 + 2 \text{ and } Q(s) = s + 4. \tag{3.12}$$

*In this case $\Omega$ contains three intervals: $\Omega_1 = (0, 1)$ (type 32), $\Omega_2 = (1, \sqrt{2})$ (type 22) and $\Omega_3 = (\sqrt{2}, 1.91]$ (type 21).*

*The stability crossing curves consist of three series of curves. Since type 21 has already been shown in Example 3.1 above, here we will only show the two series corresponding to $\Omega_2$ and $\Omega_1$. The series corresponding to $\Omega_2$ of type 22 is shown in Figure 3.3 (right). We can see that both ends approach infinity along the horizontal direction. The series corresponding to $\Omega_1$ of type 32 is shown in Figure 3.4. The curves start from infinite in directions that can be calculated by (3.11), and end at infinity along the horizontal direction.*
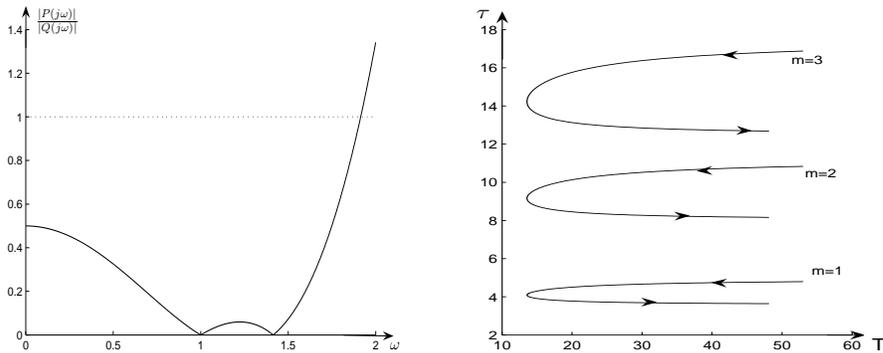


FIG. 3.3. *Example 3.2. Left: $|P(j\omega)|/|Q(j\omega)|$ against $\omega$. Right: $\mathcal{T}_2^m$, $m = 1, 2, 3$ (type 22).*

REMARK 1. *Starting from practical models encountered in the literature, we will illustrate other types in Section 4. More precisely, the crossing sets of the examples*
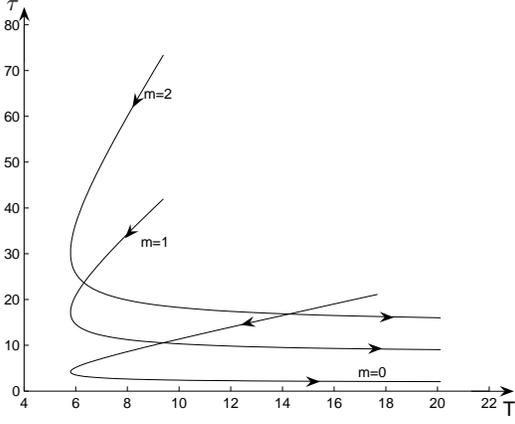
FIG. 3.4. *Example 3.2.* $\mathcal{T}_2^m$, $m = 0, 1, 2$ *(type 32)*

*we consider include intervals of the* type 31 *(linearized Cushing equation, simplified helicopter model),* type 12 *or* type 21 *(second-order example, simplified traffic flow model).*

**3.3. Tangents and smoothness.** For a given $k$ we will discuss the smoothness of the curves in $\mathcal{T}_k^m$ and thus of

$$\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}_k = \bigcup_{k=1}^{N} \bigcup_{m=-\infty}^{+\infty} \left( \mathcal{T}_k^m \bigcap \mathbb{R}_+^2 \right).$$

In this part we use an approach based on the implicit function theorem.

For this purpose, we consider $T$ and $\tau$ as implicit functions of $s = j\omega$ defined by (2.1). For a given $m$ and $k$, as $s = j\omega$ moves along the imaginary axis with $\omega \in \Omega_k$, $(T, \tau) = (T(\omega), \tau(\omega))$ moves along $\mathcal{T}_k^m$. For a given $\omega \in \Omega_k$, let:

$$R_0 = Re\left( \frac{j}{s} \frac{\partial D(s, T, \tau)}{\partial s} \right)_{s=j\omega}$$

$$= \frac{1}{\omega} Re\left\{ [nTP(j\omega) + (1 + j\omega T)P'(j\omega)] \right.$$
$$\left. \cdot (1 + j\omega T)^{n-1} + (Q'(j\omega) - \tau Q(j\omega))e^{-j\omega\tau} \right\},$$

$$I_0 = Im\left( \frac{j}{s} \frac{\partial D(s, T, \tau)}{\partial s} \right)_{s=j\omega}$$

$$= \frac{1}{\omega} Im\left\{ [nTP(j\omega) + (1 + j\omega T)P'(j\omega)] \right.$$
$$\left. \cdot (1 + j\omega T)^{n-1} + (Q'(j\omega) - \tau Q(j\omega))e^{-j\omega\tau} \right\},$$

$$R_1 = Re\left( \frac{1}{s} \frac{\partial D(s, T, \tau)}{\partial T} \right)_{s=j\omega} = Re\left( n(1 + j\omega T)^{n-1} P(j\omega) \right),$$

$$I_1 = Im\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial T}\right)_{s=j\omega} = Im\left(n(1+j\omega T)^{n-1}P(j\omega)\right),$$

$$R_2 = Re\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial \tau}\right)_{s=j\omega} = -Re\left(Q(j\omega)e^{-j\omega\tau}\right),$$

$$I_2 = Im\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial \tau}\right)_{s=j\omega} = -Im\left(Q(j\omega)e^{-j\omega\tau}\right).$$

Then, since $D(s;\ T,\tau)$ is an analytic function of $s, T$ and $\tau$, the implicit function theorem indicates that the tangent of $\mathcal{T}_k^m$ can be expressed as

$$\begin{pmatrix}\frac{dT}{d\omega} \\ \frac{d\tau}{d\omega}\end{pmatrix} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1}\begin{pmatrix} R_0 \\ I_0 \end{pmatrix}$$

$$= \frac{1}{R_1I_2 - R_2I_1}\begin{pmatrix} R_0I_2 - I_0R_2 \\ I_0R_1 - R_0I_1 \end{pmatrix}, \tag{3.13}$$

provided that

$$R_1I_2 - R_2I_1 \neq 0, \tag{3.14}$$

and $dT/d\omega$ and $d\tau/d\omega$ do not vanish simultaneously.

It follows that $\mathcal{T}_k$ is smooth everywhere except possibly at the points where either

$$R_1I_2 - R_2I_1 = 0, \tag{3.15}$$

or when

$$\frac{dT}{d\omega} = \frac{d\tau}{d\omega} = 0. \tag{3.16}$$

From the above discussions, we can conclude the following Proposition.

PROPOSITION 3.2. *The curve $\mathcal{T}_k^m$ is smooth everywhere except possibly at the points corresponding to $s = j\omega$ in either of the following two cases:*

*1) $s = j\omega$ is a multiple solution of (2.1);*

*2) $\omega$ is a type 1 end point of $\Omega_k$.*

**Proof.** From the above discussion, we only need to show that (3.15) or (3.16) can be satisfied only in the above two cases.

If (3.16) is satisfied then, in view of (3.13), $R_0 = I_0 = 0$, which implies

$$\frac{\partial D}{\partial s} = 0.$$

This, together with $D = 0$, means that $s = j\omega$ is a multiple solution of (2.1) in case 1) above.

If Condition (3.15) is satisfied, then

$$\frac{I_1}{R_1} = \frac{I_2}{R_2},$$

or

$$\angle\left(n(1+j\omega T)^{n-1}P(j\omega)\right) = \angle\left(-Q(j\omega)e^{-j\omega\tau}\right).$$

But (2.1) implies

$$\angle\left((1+j\omega T)^n P(j\omega)\right) = \angle\left(-Q(j\omega)e^{-j\omega\tau}\right).$$

Therefore, $\angle(1+j\omega T) = 0$, which in turn means $T = 0$. From this, we can conclude $|P(j\omega)| = |Q(j\omega)|$, and $\omega$ is a type 1 end point of $\Omega_k$. $\square$

**3.4. Direction of crossing.** Next we will discuss the direction in which the solutions of (2.1) cross the imaginary axis as $(T, \tau)$ deviates from the curve $\mathcal{T}_k^m$. We will call the direction of the curve that corresponds to increasing $\omega$ the *positive direction*. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*.

To establish the direction of crossing we need to consider $T$ and $\tau$ as functions of $s = \sigma + j\omega$, i.e., functions of two real variables $\sigma$ and $\omega$, and partial derivative notation needs to be adopted. Since the tangent of $\mathcal{T}_k^m$ along the positive direction is $\left( \dfrac{\partial T}{\partial \omega}, \dfrac{\partial \tau}{\partial \omega} \right)$, the normal to $\mathcal{T}_k^m$ pointing to the left hand side of positive direction is $\left( -\dfrac{\partial \tau}{\partial \omega}, \dfrac{\partial T}{\partial \omega} \right)$. Corresponding to a pair of complex conjugate solutions of (2.1) crossing the imaginary axis along the horizontal direction, $(T, \tau)$ moves along the direction $\left( \dfrac{\partial T}{\partial \sigma}, \dfrac{\partial \tau}{\partial \sigma} \right)$. So, as $(T, \tau)$ crosses the stability crossing curves from the right hand side to the left hand side, a pair of complex conjugate solutions of (2.1) cross the imaginary axis to the right half plane, if

$$\left( \frac{\partial T}{\partial \omega}\frac{\partial \tau}{\partial \sigma} - \frac{\partial \tau}{\partial \omega}\frac{\partial T}{\partial \sigma} \right)_{s=j\omega} > 0, \tag{3.17}$$

i.e. the region on the left of $\mathcal{T}_k^m$ gains two solutions on the right half plane. If the inequality (3.17) is reversed then the region on the left of $\mathcal{T}_k^m$ loses two right half plane solutions. Similar to (3.13), we can express

$$\left( \begin{array}{c} \frac{\partial T}{\partial \sigma} \\ \frac{\partial \tau}{\partial \sigma} \end{array} \right)_{s=j\omega} = \frac{1}{R_1 I_2 - R_2 I_1} \left( \begin{array}{c} R_0 R_2 + I_0 I_2 \\ -R_0 R_1 - I_0 I_1 \end{array} \right). \tag{3.18}$$

Using this we arrive at the following Proposition.

PROPOSITION 3.3. *Let $\omega \in (\omega_k^\ell, \omega_k^r)$ and $(T, \tau) \in T_k$ such that $j\omega$ is a simple solution of the characteristic equation*

$$D(s;\ T, \tau) = 0,$$

*given by (2.1) and*

$$D(j\omega';\ T, \tau) \neq 0, \qquad \forall \omega' > 0, \quad \omega' \neq \omega$$

*(i.e. $(T, \tau)$ is not an intersection point of two curves or different sections of a single curve of $T$).*

*Then, as $(T, \tau)$ crosses the stability crossing curves from the right-hand side to the left-hand side at this point, a pair of solutions of (2.1) cross the imaginary axis to the right, through $s = \pm j\omega$ if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

**Proof.** Direct computation shows that

$$\left( \frac{\partial T}{\partial \omega}\frac{\partial \tau}{\partial \sigma} - \frac{\partial \tau}{\partial \omega}\frac{\partial T}{\partial \sigma} \right)_{s=j\omega} = \frac{(R_0^2 + I_0^2)(R_2 I_1 - R_1 I_2)}{(R_1 I_2 - R_2 I_1)^2}$$

Therefore (3.17) can be written as $R_2 I_1 - R_1 I_2 > 0$. $\qquad\qquad\square$

**4. Illustrative Examples.** In order to illustrate the cases presented in the previous sections, we shall consider four examples: the linearized Cushing equation with a gap (first-order system), a simplified helicopter model (second-order system), a second-order system encountered in control engineering and finally a simplified traffic flow model.

EXAMPLE 4.1 (linearized Cushing equation with a gap). *Cushing has formulated and analyzed some general population growth models [5], and some of them have been largely treated in the literature (see, for instance, [1, 2, 4, 15], and the references therein). One of these models leads to the following characteristic equation* $(s + \alpha)(1 + sT)^n + \beta e^{-s\tau} = 0$, *where $\alpha$ is the death rate per unit time, and $\beta$ is a constant corresponding to the maternity function. Based on the particular form of the characteristic equation, it is easy to see that the only interesting case is $|\alpha| < |\beta|$. Otherwise, the crossing set $\Omega$ is empty.*

*If $|\alpha| < |\beta|$, then $\Omega = \Omega_1 = (0, \sqrt{\beta^2 - \alpha^2}]$, which is of type 31. The corresponding pairs $(T, \tau)$ are given by:*

$$T = \frac{1}{\omega} \left[ \left( \frac{\beta^2}{\omega^2 + \alpha^2} \right)^{1/n} - 1 \right]^{1/2},$$

$$\tau_m = \frac{1}{\omega} \left[ \angle \left( \frac{-\beta}{(\alpha + j\omega)(1 + j\omega T)^n} \right) + 2m\pi \right].$$

*According to Proposition 3.2, we get:*

$$\lim_{\omega \to \sqrt{\beta^2 - \alpha^2}} T = 0, \quad \lim_{\omega \to 0} T = \infty, \quad \lim_{\omega \to 0} \tau_m = \infty$$

*and*

$$\lim_{\omega \to \sqrt{\beta^2 - \alpha^2}} \tau_m = \frac{1}{\sqrt{\beta^2 - \alpha^2}} \left( 2m\pi + \angle \left( \frac{-\beta}{\alpha} \right) - \arctan \frac{\sqrt{\beta^2 - \alpha^2}}{\alpha} \right).$$

*Also the slopes of the corresponding asymptotes are given by*

$$\lim_{\omega \to 0} \frac{\tau}{T} = \frac{-n \arctan \left[ \left( \frac{\beta^2}{\alpha^2} \right)^{1/n} - 1 \right]^{1/2} + \angle \left( \frac{-\beta}{\alpha} \right) + 2m\pi}{\left[ \left( \frac{\beta^2}{\alpha^2} \right)^{1/n} - 1 \right]^{1/2}}$$

*Figures 4.1 (right) and 4.2 plot $\tau_m$, $m \in \{0, 1, 2, 3, 4\}$ against $T$ in the case $n = 1$ and $n = 4$ for $\alpha = 3$ and $\beta = 5$, respectively. The crossing set $\Omega = (0, 4]$ is shown in figure 4.1 (left). We observe that $\tau_{m+1}(\omega) > \tau_m(\omega)$, $\forall m \geq 0$ and $\omega \in \Omega$. Furthermore, it is easy to see that at $\omega = 2 \in \Omega$, for any $m$,*

$$R_2 I_1 - R_1 I_2 = -13n \left( \frac{25}{13} \right)^{\frac{n-1}{n}} \left[ \left( \frac{25}{13} \right)^{1/n} - 1 \right]^{1/2} < 0.$$

*Therefore, we can conclude that as $\tau$ increases from zero, every time it crosses the stability crossing curve in Figure 4.1 (right) or 4.2, the equation (2.1) gains two*
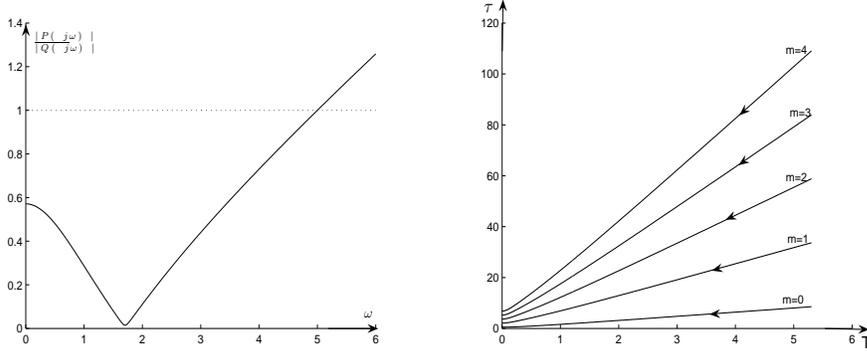
FIG. 4.1. *Linearized Cushing equation example.*
*Left: $|P(j\omega)|/|Q(j\omega)|$ against $\omega$. Right: $\tau_m$, $m \in \{0, 1, 2, 3, 4\}$ versus $T$ when $n = 1$*
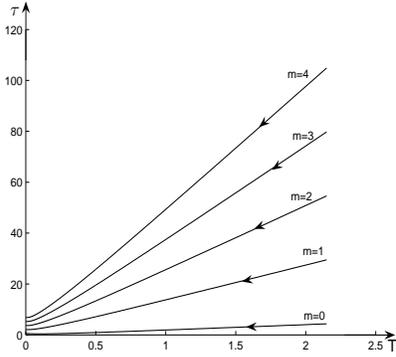


FIG. 4.2. *Linearized Cushing equation example. $\tau_m$, $m \in \{0, 1, 2, 3, 4\}$ versus $T$ when $n = 4$*

*additional right half plane solutions. In addition, we can easily see that the system is stable when $T = 0$, $\tau = 0$. Therefore, the linearized Cushing equation is stable only in the region below the curve labelled "$m = 0$" and above the $T$-axis.*

EXAMPLE 4.2 (Simplified helicopter model). *Consider a helicopter model [18, 19] consisting of a fixed base and a rotary arm mounted on the base. The arm carries the helicopter body on one end, and a counterweight on the other. The arm can make an elevation motion around an angle $x$. The corresponding nonlinear mathematical model is*

$$J \cdot \ddot{x} = -g \cdot y \cdot (M + m) \cdot \sin x + 2 \cdot k_t \cdot r \cdot v(t) \qquad (4.1)$$

*where $k_t, g$ represent the motor and the gravity constants, $y$ is the distance between the rotation point and the rotary arm, $r$ is the distance from the helicopter body to the fixed base, $m$ and $M$ denote the mass of the helicopter blades (including the motors and the fixing devices), and the counterweight, respectively. $J$ is the moment of inertia around the rotating point, and $v(t)$ the corresponding voltage. We note that all of these values can be explicitly measured. Linearizing around the quiescent point, one gets*

$$J \cdot \ddot{x} = -g \cdot y \cdot (M + m) \cdot x + 2 \cdot k_t \cdot r \cdot v \qquad (4.2)$$

*and finally, after the damping factor identification, we obtain the helicopter transfer function:*

$$G(s) = \frac{0.2607}{s^2 + 0.07441s + 2.904} \tag{4.3}$$

*Considering a simple PD-controller (for improving the system response of the above helicopter laboratory experiment) $G_c(s) = (16.5s + 19.5)$ with a gamma-distribution with a gap $e^{-s\tau}/(1+sT)^n$ modeling the overall communication delay, one obtains the closed-loop characteristic equation given by:*

$$(s^2 + 0.07441s + 2.904)(1 + sT)^n + (4.3015s + 5.0836)e^{-s\tau} = 0 \tag{4.4}$$

*The crossing set $\Omega$ consists of one interval $(0, 5.0002]$ of type 31 (figure 4.3 (left)). Some stability crossing curves are plotted in figure 4.3 (right).*
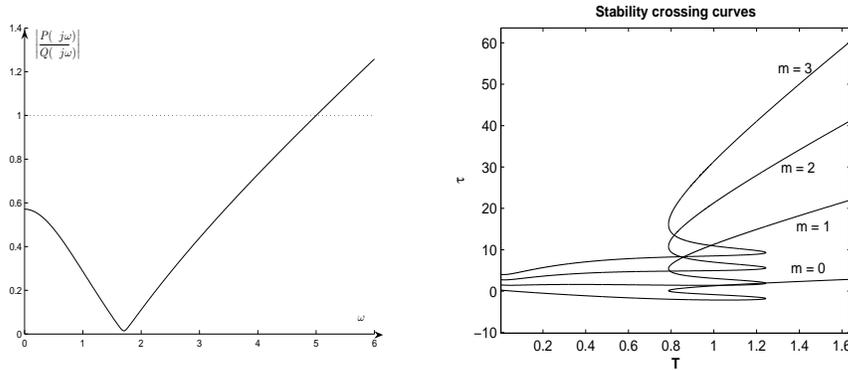


FIG. 4.3. *Simplified helicopter model: $\mathcal{T}_1^m$, $m = 0, 1, 2, 3$ for the equation (4.4).*

EXAMPLE 4.3 (Controlling second-order systems). *Consider the equation (2.1) when $Q(s) = k_1 s + k_2$ and $P(s) = s^2 + 2$. It is easy to see that if $k_1 = 0$, and $T = 0$, the characteristic equation corresponds to the closed-loop system of a simple oscillator $1/(s^2 + 2)$ controlled by a delayed output feedback of the form $k_2 e^{-s\tau}$, that is:*

$$\ddot{y}(t) + 2y(t) = u(t),$$

*with*

$$u(t) = -k_2 y(t - \tau).$$

*It is important to point out that for very small delay values $\tau$, and very small gains $k_2$, the closed-loop system is asymptotically stable, but it is not asymptotically stable if the delay $\tau$ is equal to 0, that is for the control law: $u(t) = -k_2 y(t)$. We have the so-called* stabilizing effect *of the* delay *(see, for instance, [16], and the references therein on stabilizing oscillations by using delayed feedback laws).*

*The more general system with the characteristic equation*

$$(s^2 + 2)(1 + sT)^n + (k_1 s + k_2)e^{-s\tau} = 0, \tag{4.5}$$

*can be analyzed as follows:*

*Case 1: If $|k_2| < 2$ then the crossing set $\Omega = [\omega_+, \omega_-] \setminus \{\sqrt{2}\}$, where*

$$\omega_\pm = \sqrt{\frac{k_1^2 + 4 \pm \sqrt{(k_1^2 + 4)^2 - 4(4 - k_2^2)}}{2}}.$$

*We note that*

$$\omega_- \le \sqrt{\frac{k_1^2 + 4 \pm \sqrt{(k_1^2 + 4)^2 - 16}}{2}} \le \sqrt{2} < \omega_+.$$

*Therefore, $\Omega$ consists of two intervals of type 12 and 21, respectively. More details can be found below where we will discuss the case $k_2 = 0$ and $k_1 = 1$.*
*Case 2: If $|k_2| \ge 2$ then the crossing set $\Omega = (0, \omega_+] \setminus \{\sqrt{2}\}$, where $\omega_+ > \sqrt{2}$ is defined above. So, the crossing set $\Omega$ consists of two intervals of type 32 and 21, respectively.*

*Next we consider the following special case: $k_1 = 1$ and $k_2 = 0$. Using (3.5) and (3.6), we compute the crossing set $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = [1, \sqrt{2})$ is of type 12, and $\Omega_2 = (\sqrt{2}, 2]$ is of type 21 (see also figure 4.4 (left)). Simple computation shows that*

$$T = \frac{1}{\omega} \sqrt{\left( \frac{\omega^2}{(2 - \omega^2)^2} \right)^{1/n} - 1}$$

*and*

$$\tau_m = \frac{1}{\omega} \left( \angle \frac{j\omega}{(2 - \omega^2)(1 + j\omega T)^n} + 2m\pi \right).$$

*According to the result of Proposition 3.2 we have $\lim_{\omega \to 1} T = 0$, $\lim_{\omega \to 2} T = 0$,*
*$\lim_{\omega \to \sqrt{2}} T = \infty$, $\lim_{\omega \to 1} \tau_m = -\frac{\pi}{2} + 2m\pi$, $\lim_{\omega \to 2} \tau_m = \frac{\pi}{4} + m\pi$ and*

$$\lim_{\omega \to \sqrt{2}-0} \tau_m = \frac{[2m - (n-1)/2]\pi}{\sqrt{2}}$$
$$\lim_{\omega \to \sqrt{2}+0} \tau_m = \frac{[2m - (n+1)/2]\pi}{\sqrt{2}}$$

*We will now calculate the direction of crossing. A direction calculation yields*

$$R_2 I_1 - R_1 I_2 = n(1 + \omega^2 T^2)^{n-1}(2 - \omega^2)^2 \, Im(1 - j\omega T)$$
$$= -n\omega T(1 + \omega^2 T^2)^{n-1}(2 - \omega^2)^2 < 0$$

*Therefore, using Proposition 3.3, we can conclude that as we cross the stability crossing curves from its right hand side to its left hand side, a pair of complex conjugate solutions of $D = 0$ cross the imaginary axis from the right half complex plane to the left half plane.*

*The computations above show us that the following inequalities holds:*

$$\tau_m(\sqrt{2} + 0) < \tau_m(\sqrt{2} - 0) < \tau_{m+1}(\sqrt{2} + 0), \quad \forall m \in \mathbb{Z}.$$
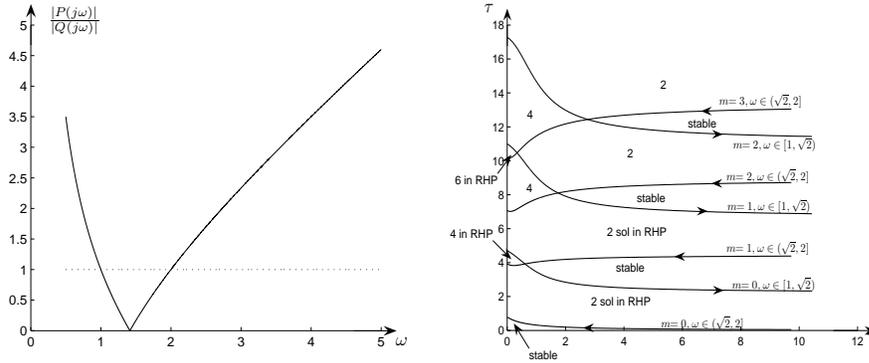
FIG. 4.4. *Second-order system example 4.3:*
*Left: $|P(j\omega)|/|Q(j\omega)|$ versus $\omega$. Right:   $\tau_m$, $m = 0, 1, 2, 3$ versus $T$ when $n = 1$.*

*This simply states that for large values of $T$ the crossing towards stability and the crossing towards instability interlace. Consider the additional fact that the system is obviously stable for $\tau = 0$ and $T = 0$, and the fact that the stability crossing curves approaches horizontal, we can conclude that the system has an* infinite number *of stable regions. Figure 4.4 shows the case when $n = 1$.*

EXAMPLE 4.4 (Traffic flow model). *Finally we consider a* time-delay microscopic system *including delayed reactions of the driver, and as explained in the introduction, we will use a* distributed delay with a gap *for modeling human driver reactions with respect to the traffic behavior. Specifically, consider the traffic flow dynamic described by (1.8) with $\alpha_1 = \alpha_2 = 2$, the stability analysis leads to the conclusion that the system has only one stability region (see figure 4.5).*
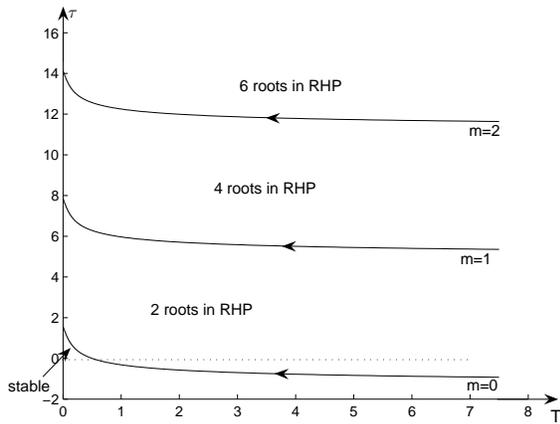


FIG. 4.5. *Traffic flow model with 2 vehicles: $\tau_m$, $m \in \{0, 1, 2\}$ versus $T$ when $n = 2$*

*More exactly, the crossing set $\Omega$ consists of one interval $(0, 4]$ and the crossing*

*curves are described by the following equations:*

$$T = \frac{1}{\omega}\sqrt{\left(\frac{16}{\omega^2}\right)^{1/n} - 1} \tag{4.6}$$

$$\tau = \frac{1}{\omega}\left(\frac{\pi}{2} - n\arctan(\omega T) + 2m\pi\right), \qquad m = 0, 1, 2, \ldots \tag{4.7}$$

*We note that Assumption V is not satisfied in this case and the shape of the crossing curves do not perfectly match with the classification proposed in section 3. However, the ideas of our approach still apply.*

*It is also important to point out that the matrix $A_1 + A_2$ that defines the characteristic equation (1.9) always has an eigenvalue at the origin. This corresponds to the situation in which the relative movement of one vehicle to the others is zero (the vehicles are either staying or moving with the same velocity).*

*On the other hand, varying n over positive integers, we can see that the stability region becomes smaller as the integer n increases (Figure 4.6).*
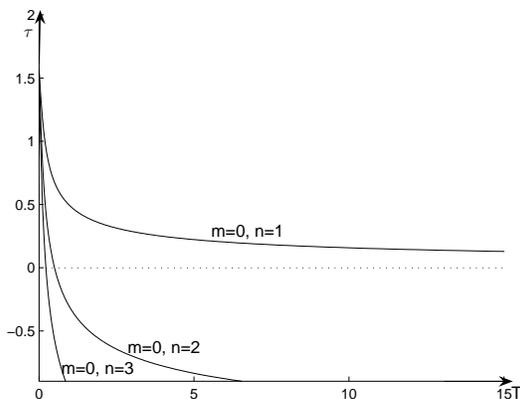


FIG. 4.6. *Traffic flow model: $\tau_1$ versus $T$ when $n \in \{1, 2, 3\}$*

**5. Concluding remarks.** This paper addressed the stability problem of shifted gamma-distributed delay systems. More specifically, we have characterized the geometry of the stability crossing curves in the parameter space defined by the gap and the corresponding mean delay. Several illustrative examples complete the presentation.

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